Math 476 - Abstract Algebra 1
Day 35 Group Assignment **Name:**______________________________

Recall: Definition 12.2 Let R be a commutative ring and let $u(x)$ and $v(x)$ be polynomials in $R[x]$. The polynomial $u(x)$ **divides** the polynomial $v(x)$ provided that there exists a polynomial $q(x) \in R[x]$ such that $v(x) = u(x)q(x)$. In this case, we say that $u(x)$ is a **factor** of $v(x)$ and sometimes write $u(x)|v(x)$.

Definition 12.8 Let F be a field and let $f(x), g(x) \in F[x]$ be polynomials that are not both zero. The **greatest common divisor** of $f(x)$ and $g(x)$ is the polynomial $d(x) \in F[x]$ that satisfies the following three conditions:

- $d(x)$ divides both $f(x)$ and $g(x)$.
- If $h(x) \in F[x]$ and $h(x)$ divides both $f(x)$ and $g(x)$, then $h(x)$ divides $d(x)$.
- $d(x)$ is a monic polynomial (that is, the leading coefficient of $d(x)$ is 1_F).

Theorem 13.3 (The Remainder Theorem) Let F be a field, let $p(x) \in F[x]$, and let $c \in F$. The remainder then $p(x)$ is divided by $(x - c)$ is equal to $p(c)$. That is, there exists a unique polynomial $q(x) \in F[x]$ such that $p(x) = (x - c)q(x) + p(c)$

Proof Sketch: By the division algorithm, there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $p(x) = (x - c)q(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(x - c)$. Since $\deg(x - c) = 1$, we must have $r(x) = 0$ or $\deg(r(x)) = 0$. In either case, $r(x) = a$ for some $a \in F$. Then $p(x) = (x - c)q(x) + a$. Therefore, $p(c) = (c - c)q(x) + a = 0_F + a = a$. Hence $p(c) = a$. Therefore, $r(x) = p(c)$. $\square$.

Definition 13.4 Let R be a commutative ring and let $p(x) \in R[x]$. An element c of R is called a **root of the polynomial** ***p(x)*** provided that $p(c) = 0_R$.

Theorem 13.5 (The Factor Theorem) Let F be a field, let $p(x) \in F[x]$, and let $c \in F$. Then c is a root of the polynomial $p(x)$ if and only if $(x - c)$ is a factor of $p(x)$.

Theorem 14.2 (Fundamental Theorem of Algebra) Every Polynomial of degree 1 or more in $\mathbb{C}[x]$ has a root in $\mathbb{C}$.

1. Let $F = \mathbb{R}$, let $f(x) = x^3 + 2x^2 - x - 2$ and let $g(x) = x^3 + x^2 - 2x$.

   (a) Show that $c = 1$ is a root of both $f(x)$ and $g(x)$. Based on this, find one common divisor of these two polynomials.

   (b) Find the greatest common divisor of $f(x)$ and $g(x)$.

   (c) Describe as clearly as you can the set of all polynomials that divide $g(x)$.

2. Let I be the subset of $\mathbb{Z}_{12}$ defined by $I = \{[0], [3], [6], [9]\} = \{k[3] : k \in \mathbb{Z}\}$.

   (a) Show that I is a subring of $\mathbb{Z}_{12}$.

(b) Show that if $[a] \in \mathbb{Z}_{12}$ and $[b] \in I$, then $[a][b] \in I$ and $[b][a] \in I$.

(c) Define a relation $\equiv_I$ on $\mathbb{Z}_{12}$ via $[a] \equiv_I [b]$ if and only if $[b] - [a] \in I$.

i. Describe all the elements of $\mathbb{Z}_{12}$ that are related to $[0]$.

ii. Describe all the elements of $\mathbb{Z}_{12}$ that are related to $[1]$.

iii. Describe all the elements of $\mathbb{Z}_{12}$ that are related to $[5]$.

Definition 16.2 An **ideal** I in a ring R is a subring of R such that $rx \in I$ and $xr \in I$ for all $r \in R$ and $x \in I$. This property if called the *absorbing property* or *closure under outside multiplication.*

Theorem 16.4(The Ideal Test). Let R be a ring. A subset I of R is an ideal of R if and only if:

- I is non-empty;
- $a - b \in I$ for every $a, b \in I$; and
- $ra \in I$ and $ar \in I$ for every $r \in R$ and $a \in I$.

3. For each of the following, use the Ideal Test to determine whether or not the set I is an ideal of the ring R.

(a) $R = \mathbb{Z}$ and $I = \{0\}$.

(b) $R = \mathbb{Z}$ and $I = \{2n : n \in \mathbb{Z}\}$.

(c) $R = \mathbb{R}$ and $I = \mathbb{Q}$.