Math 476 - Abstract Algebra 1
Day 37 Group Assignment **Name:**________________________________

Recall: Definition 16.36 Let R and S be rings. A function $\varphi : R \to S$ is a **homomorphism** of rings if $\varphi(a+b) = \varphi(a)+\varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

Theorem 16.39 Let $\varphi : R \to S$ be a ring homomorphism. Let 0_R and 0_S be the additive identities in R and S, respectively.

1. $\varphi(0_R) = 0_S$.
2. If $a \in R$, then $-\varphi(a) = \varphi(-a)$.
3. If $a, b \in R$, then $\varphi(a - b) = \varphi(a) - \varphi(b)$.
4. If R has identity 1_R, φ is an epimorphism, and S is not the trivial ring, then S has identity 1_S and $\varphi(1_R) = 1_S$.
5. If R has identity 1_R, φ is an epimorphism, and S is not the trivial ring, then $\varphi(u)$ is a unit in S for any unit u in R, and $(\varphi(u))^{-1} = \varphi(u^{-1}$.

Proof: Parts (a), (b), and (c) are presentation problem eligible. The proofs of (d) and (e) can be found on p. 229 on your textbook.

Definition 16.40 Let $\varphi : R \to S$ be a homomorphism of rings. The **kernel** of φ is the set $Ker(\varphi) = \{r \in R : \varphi(r) = 0_S\}$.

Definition 16.44 Let $\varphi : R \to S$ be a homomorphism of rings. The **image** of φ is the set $Im(\varphi) = \{\varphi(r) : r \in R\}$.

1. Let $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_6$ be defined by $\varphi([k]_{12}) = [4k]_6$.
   (a) Prove that φ is a ring homomorphism.

   (b) Find $Ker(\varphi)$.

   (c) Show that $Ker(\varphi)$ is an ideal of $\mathbb{Z}_{12}$.

   (d) Find $Im(\varphi)$.

   (e) Show that $Im(\varphi)$ is a subring of $\mathbb{Z}_6$.

Theorem 16.42 If $\varphi : R \to S$ is a ring homomorphism, then $Ker(\varphi)$ is an ideal of R.

Proof: Presentation Eligible Proof.

Theorem 16.43 Let $\varphi : R \to S$ be a ring homomorphism, then φ is a monomorphism if and only if $Ker(\varphi) = \{0_R\}$, where 0_R is the additive identity in R.

Proof: Let $\varphi : R \to S$ be a ring homomorphism and let 0_R and 0_S be the additive identities in R and S, respectively. To show the forward direction, suppose that φ is a monomorphism. By Theorem 16.39, $\varphi(0_R) = 0_S$, so 0_R is in $Ker(\varphi)$. Suppose $k \in Ker(\varphi)$. Then $\varphi(k) = 0_S = \varphi(0_R)$. Since φ is a monomorphism, it is 1-1, hence $k = 0_R$. Thus $Ker(\varphi) = \{0_R\}$.

To show the reverse implication, suppose that $Ker(\varphi) = \{0_R\}$. Let $a, b \in R$ be ring elements such that $\varphi(a) = \varphi(b)$. Then, again by Theorem 16.39, $\varphi(a - b) = \varphi(a) - \varphi(b) = 0_S$. Thus $a - b \in Ker(\varphi)$. Since $Ker(\varphi) = \{0_R\}$, we must have $a - b = 0_R$, this $a = b$. Hence φ is a monomorphism. $\square$.

Theorem 16.46 If $\varphi : R \to S$ is a ring homomorphism, then $Im(\varphi)$ is a subring of S.

Proof: Presentation Eligible Proof.

Theorem 16.48(First Isomorphism Theorem for Rings). Let $\varphi : R \to S$ be a ring homomorphism. Then $R/Ker(\varphi) \cong Im(\varphi)$.

Proof: See Textbook p. 232

Definition 16.26 An ideal I in a commutative ring R is a **maximal ideal** if $I \neq R$ and there is no proper ideal J in R such that $I \subset J$.

Theorem 16.27 Let R be a commutative ring with identity and I an ideal of R. Then R/I is a field if and only if I is a maximal ideal.

Proof: See pages 225-226 in your textbook.

Definition 16.32 A proper ideal I in a commutative ring R is a **prime ideal** if for any $a, b \in R$, whenever $ab \in I$, then $a \in I$ or $b \in I$.

Euclid's Lemma Let p be any prime number. Then $\langle p \rangle$ is a prime ideal of $\mathbb{Z}$.

Theorem 16.34 Let R be a commutative ring with identity and I a proper ideal of R. Then R/I is an integral domain if and only if I is a prime ideal.

2. (a) Find all of the ideals of $\mathbb{Z}_{12}$. Be sure to explain how you know your list is complete.

   (b) Which ideals of $\mathbb{Z}_{12}$ are maximal ideals? Justify your answer.

   (c) Which ideals of $\mathbb{Z}_{12}$ are prime ideals? Justify your answer.