**Intro to Groups**

**Recall:** Back in I-1, we were introduced to the Axioms of Integer Arithmetic. Before moving on, turn to p. 5 in your textbook and review these axioms. Focus mainly on the axioms that involve *only* the addition operation.

**Definition 20.2** A **group** is a set $G$ on which one binary operation, denoted $\cdot$ is defined such that all of the following axioms hold:

- The set $G$ is **closed under its operation** [$a \cdot b \in G$ for all $a, b \in G$].
- The operation $\cdot$ is **associative in** $G$ [$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$].
- The set $G$ **contains an identity element** [there exists an element $e \in G$ such that $a \cdot e = a = e \cdot a$ for all $a \in G$].
- **The set $G$ contains an inverse for each of its elements** [for each $a \in G$ there exists an element $b \in G$ such that $a \cdot b = e = b \cdot a$].

**Note:** The operation in the definition above is **not** required to be commutative (it may be in some groups but not in others). Because of this, we give the following definition.

**Definition 20.3** A group $G$ is an **Abelian group** if $a \cdot b = b \cdot a$ for all $a, b \in G$ (i.e. the operation $\cdot$ is commutative in $G$).

- An Abelian group is sometimes called a *commutative group.*
- In some cases, we will use $+$ rather than $\cdot$ to denote the operation in a group (especially in cases when that notation is the usual way of denoting the operation and avoids unnecessary confusion, such as for the integers under addition).

1. Which of the following are examples of groups. You should take a moment to determine whether or not the four group axioms are satisfied. In particular, identify the identity element and determine if each element has an inverse. For those that are groups, determine whether or not the group is Abelian.

   (a) The set of symmetries of an object with composition as the operation.

   (b) $\mathbb{Z}_n$ under the operation addition

   (c) $\mathbb{Z}_n$ under the operation multiplication

   (d) $\mathcal{M}_{n\times n}\mathbb{R}$, the set of $n \times n$ matrices with real entries under matrix addition.

   (e) $\mathcal{M}_{n\times n}\mathbb{R}$, the set of $n \times n$ matrices with real entries under matrix multiplication

   (f) The set $\{-1, 1\}$ under the operation multiplication.

   (g) $\mathbb{R}^+$ the set of all positive real numbers under multiplication.

   (h) $\mathcal{B}$, the set of all bijections mapping the real numbers to the real numbers under the operation function composition.

2. Let $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ under the operation defined via the table given below.

| $\circ$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
| $a_2$ | $a_2$ | $a_1$ | $a_5$ | $a_6$ | $a_3$ | $a_4$ |
| $a_3$ | $a_3$ | $a_6$ | $a_1$ | $a_5$ | $a_4$ | $a_2$ |
| $a_4$ | $a_4$ | $a_5$ | $a_6$ | $a_1$ | $a_2$ | $a_3$ |
| $a_5$ | $a_5$ | $a_4$ | $a_2$ | $a_3$ | $a_6$ | $a_1$ |
| $a_6$ | $a_6$ | $a_3$ | $a_4$ | $a_2$ | $a_1$ | $a_5$ |

(a) Determine whether or not the set $S$ is closed under the given operation. Justify your answer.

(b) Determine whether or not there is an identity operation. Justify your answer.

(c) For each element that has one, identify its inverse element. Justify your answer.

(d) Proving that an operation that is given by a table is associative is difficult (mainly due to the number of cases you would need to check). Verify one case by showing that $a_2(a_3a_4) = (a_2a_3)a_4$.

3. For each of the following groups, determine: the identity element, the inverse of an arbitrary element, and determine whether or not the group is commutative.

(a) $\mathbb{R}^+$ the set of all positive real numbers under multiplication.

(b) $\mathcal{B}$, the set of all bijections mapping the real numbers to the real numbers under the operation function composition.

(c) $GL_2\mathbb{R}$, the set of all invertible $2 \times 2$ matrices with real entries under matrix multiplication

**Theorem 20.7** (Group Cancellation Law). Let $G$ be a group, and let $a, b, c \in G$. If $ac = bc$, then $a = b$. Similarly, if $ab = ac$ then $b = c$.

Read and make sure that you understand the proof of this theorem on page 287 in your textbook.