Math 476 - Abstract Algebra 1
Day 9 Group Assignment

Name:______________________________________

The Order of a Group

Recall: Definition 20.2 A **group** is a set G on which one binary operation, denoted $\cdot$ is defined such that all of the following axioms hold:

- The set G is **closed under its operation** [$a \cdot b \in G$ for all $a, b \in G$].
- The operation $\cdot$ is **associative in** G [$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$].
- The set G **contains an identity element** [there exists an element $e \in G$ such that $a \cdot e = a = e \cdot a$ for all $a \in G$].
- **The set** G **contains an inverse for each of its elements** [for each $a \in G$ there exists an element $b \in G$ such that $a \cdot b = e = b \cdot a$].

1. Let G be a group. Prove that $\left(a^{-1}\right)^{-1} = a$.

2. Determine which of the following statements hold in **any** group G. Justify your answer.
   - $(ab)^{-1} = a^{-1}b^{-1}$
   - $(ab)^{-1} = b^{-1}a^{-1}$

Definition 20.10 Let G be a group. If G contains only a finite number of elements, then G has **finite order** and we say $G$is a **finite group**. If G contains exactly m distinct elements, then the **order** of G, denoted $|G|$, is m. If G contains infinitely many elements, then G has **infinite order** and we say G is an **infinite group**.

3. Give an example of a finite group (be sure to identify its order) and an example of an infinite group.

4. Consider the groups $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, and $\mathbb{Z}_8$ (under the operation of multiplication on congruence classes).

(a) Find the units in each of these groups.

(b) Let U_n represent the set of units in $\mathbb{Z}_n$. Construct multiplication tables for U_4, U_5, U_6, and U_8.

(c) Based on these tables, which of these sets are groups under multiplication?

Theorem 20.15 Let S be a set on which an associative binary operation of multiplication is defined such that S contains an identity element 1_S. Let $U(S)$ be the set of units in S. Then $U(S)$ is a group under the operation of multiplication.