Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 11 **Name:**________________________________
Due: At the end of class on Tuesday, October 9th

Theorem 3.2.1 (The Principle of Strong Mathematical Induction [PMI])
Let $P(n)$ be a statement about the positive integer n, so that n is a free variable in $P(n)$. Suppose the following:

- (PSMI 1) The statement $P(1)$ is true.
- (PSMI 2) If for all positive integers k with $1 \leq k \leq m$, if $P(m)$ is true, then $P(m+1)$ is true.

Then, for all positive integers n, the statement $P(n)$ is true.

1. In your own words, explain how the Principle of Strong Mathematical Induction differs from the Principle of Mathematical Induction. In particular, contrast which previous cases you are allowed to assume are true when trying to prove that $P(m+1)$ is true.

2. Make a reasonable conjecture about the value of this expression for an unspecified value of n (your answer should be a formula given in terms of n).

   Example: (The chocolate bar problem) Suppose we have a chocolate bar consisting of n rows, each containing m squares of chocolate. How many "cuts" are needed to break the entire bar down into individual squares? A "cut" consists of breaking one seam (an entire row or an entire column) in a single connected piece of the original bar.

3. Consider the sequence of numbers defined as follows: $a_1 = 3$, $a_2 = 4$, and $a_{n+1} = \frac{1}{3}(2a_n + a_{n-1})$. Use strong induction to prove that for all positive integers n, $3 \leq a_n \leq 4$.

Recall: Theorem 3.2.3 (Fundamental Theorem of Arithmetic). Every positive integer $n > 1$ can be written as a product of primes; i.e., for every positive integer $n > 1$, there exists $s \in \mathbb{Z}^+$ and primes $p_1, p_2, \cdots, p_s$ such that $n = p_1 p_2 \cdots p_s$.

Proof: We will use strong induction on n, where $n \geq 2$.

Base Case: Note that 2 is prime, hence a can be written as $2 = p_1$, where $p_1 = 2$ is prime.

Inductive Step: Let $m \in \mathbb{Z}$ with $m \geq 2$ and assume that for all integers k with $2 \leq k < m$, k is a product of primes. We must prove that $m + 1$ is also a product of primes.

Case 1: $m + 1$ is prime.

Then $m + 1 = p_1$ for a single prime, and hence is a product of primes.

Case 2: $m + 1$ is not prime.

Then, by Definition 2.1.7, there exist $a, b \in \mathbb{Z}^+$ such that $m + 1 = ab$, with $a \neq 1$ and $b \neq 1$.

Notice that we must have $1 < a < m + 1$, so $2 \leq a \leq m$, and, similarly, $2 \leq b \leq m$.

Then we may apply the strong induction hypothesis to conclude that $a = p_1 p_2 \cdots p_i$ and $b = q_1 q_2 \cdots q_j$ where $i, j \geq 1$ and $p_1, p_2, \cdots, p_i, q_1, q_1, \cdots, q_j$ are all primes.

Hence $ab = p_1 p_2 \cdots p_i \cdot q_1 q_2 \cdots q_j$, a product of primes. Hence, by PSMI, every positive integer $n > 1$ is a product of primes.

4. **Find and clearly describe the flaw in the reasoning of the following "proof":**

Let $P(n)$ represent the statement "For every set of n horses, all the horses in the set are the same color.".

Note that $P(1)$ is true since in any set containing a single horse, all the horses in the set are the same color.

Let $m > 1$ and suppose that $P(m)$ is true. That is, suppose that in any set of exactly m horses, all of the horses in the set are the same color.

Consider a set of $m + 1$ horses. Taken in order, the set consists of horses $h_1, h_2, \cdots h_m, h_{m+1}$. Notice that the subset $h_1, h_2, \cdots h_m$ (all except the last horse) is a set of exactly m horses, so, by the induction hypothesis, all of these horses are the same color. Similarly, the set $h_1, h_3, \cdots h_m, h_{m+1}$ (all except the second horse) is also a set of exactly m horses, so all of these horses are the same color.

Notice that in both cases, all of the horses share the same color as h_1, the first horse. Therefore, all of the horses, $h_1, h_2, \cdots h_m, h_{m+1}$ are the same color as h_1, so they are all the same color. Hence $P(m + 1)$ is true.

Hence, by PMI, all horses are the same color.

5. The remainder of this handout consists of general presentation problems that do not need to be handed in for grading but that you can present on the board during class this week.

   (a) Use the Principle of Mathematical Induction to prove that for all $n \geq 1$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

   (b) Use the Principle of Mathematical Induction to prove that for all $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

   (c) Use the Principle of Mathematical Induction to prove that for all $n \geq 1$, $4^n - 1$ is divisible by 3.

   (d) Use the Principle of Mathematical Induction to prove that for all $n \geq 5$, $n^2 < 2^n$.

   (e) Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$

   (f) Prove or disprove: Every non-negative integer can be written as the sum of at most 3 perfect squares.

   (g) Prove or disprove: Let a, b, and c be integers. If $a|b$ and $a|c$, then $a|(b+c)$.

   (h) Prove or disprove: Let a, b, c and d be integers. If $a|b$ and $c|d$, then $ac|bd$.

   (i) Prove that $x^2 + y^2 = 11$ has no integer solutions.

   (j) Prove or disprove: If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

   (k) Prove or disprove: If a does not divide bc, then a does not divide b.

   (l) Formulate a conjecture about the decimal digits that appear as the final digit of the fourth power of an integer. Prove your conjecture using proof by cases.

   (m) Suppose that a fast food restaurant sells chicken nuggets in packs of 4, 7, or 9. What is the largest number of chicken nuggets that you **cannot** buy exactly (Fully justify your answer).

   (n) Suppose that a different restaurant sells chicken nuggets in packs of 4 or 15. What is the largest number of chicken nuggets that you **cannot** buy exactly (Fully justify your answer).