Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 12 **Name:**________________________________
Due: At the end of class on Thursday, October 11th

The Language of Sets: Sets are one of the most important and and fundamental mathematical objects. The goal of this chapter of content is to help familiarize you with sets and their properties. You have probably already encountered sets sometime in your previous mathematical experience. For now, we will consider "sets" as a fundamental concept to be an undefined term. We think of them as a "collection of objects". In order for a set to be *well-defined*, membership in the set must be decidable. That is, there should be a clear process or criterion that we can use to determine whether or not any *element* taken from the underlying universe is in the set (or is not in the set).

Notation:

- Given a set A, we use $x \in A$ to denote "x is an element of the set A", and we write $x \notin A$ to denote "x is not an element of the set A".
- There are several ways that we will use to describe sets. One nice way is to use *roster notation*. In this method, we "list" the elements of the set inside a pair of curly braces. For example:

  $A = \{a, e, i, o, u\}$
  $B = \{2, 4, 6, 8\}$
  $P = \{2, 3, 5, 7, 11, \cdots\}$

  Note that while the first two examples are precise, the last example is imprecise, as the reader needs to guess a rule that determines membership in order to decide whether particular elements are (or are not) in the set P. Also note that we should specify the underlying universe for these sets (here, we take the universe to be $\mathbb{N}$).
- A more precise way to describe sets is to use "set builder" notation. In this method, we use a variable (referring to the underlying universe) along with a definition or rule for the set. More formally, set set $A = \{x \in X | P(x)\}$ or, equivalently, $A = \{x \in X : P(x)\}$, where X is the underlying universe, and $P(x)$ is a statement with a single free variable x. The set A consists of all of the elements of X for which the statement $P(x)$ is true.

  $A = \{\ell \in \mathcal{L} : \ell$ is always a vowel $\}$, where $\mathcal{L}$ is the lowercase roman alphabet.
  $B = \{n \in \mathbb{N} | \, n$ is a one-digit even number $\}$.
  $P = \{p \in \mathbb{N} : p$ is prime $\}$.

1. Think of an example of a set given in roster notation and an example of a set given in set builder notation.

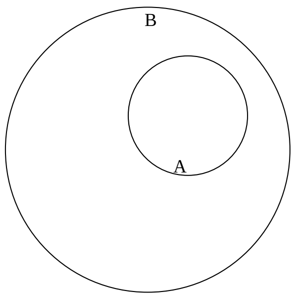
   Note: One important set is the unique set with no elements. This set is called the **empty set** and is denoted by $\emptyset$. Using roster notation, $\emptyset = \{ \ \}$.

2. It is important to notice that $\emptyset \neq \{\emptyset\}$. In your own words, describe $\{\emptyset\}$ and explain why it differs from $\emptyset$.

Definition 4.1.4: Let A and B be sets. Then A is a subset of B (denoted $A \subseteq B$) if every element of A is also an element of B. More formally, we say $A \subseteq B$ if $(\forall x)[x \in A \Rightarrow x \in B]$.

When this fails to hold, we write $A \not\subseteq B$ to indicate that A is not a subset of B.

One nice way of illustrating relationships between sets is to use **Venn diagrams**. Here is a Venn diagram illustrating $A \subseteq B$.



3. Given examples of three sets A, B, and C with $A \subseteq B$ and $A \not\subseteq C$

4. Consider the sets $A = \{n \in \mathbb{Z} : (\exists k \in \mathbb{Z})[n = 6k + 2]$ and $B = \{n \in \mathbb{Z} : n \text{ is even } \}$. Prove that $A \subseteq B$ but $B \not\subseteq A$.

Definition 4.1.6 Let A and B be sets. Then $A = B$ is $(\forall x)[x \in A \Leftrightarrow x \in B]$.
Equivalently, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

5. Let $A = \{n \in \mathbb{Z} : n + 3 \text{ is even }\}$ and $B = \{n \in \mathbb{Z} : n \text{ is odd }\}$. Prove that $A = B$.

Theorem 4.1.8: Let A, B, and C be sets. $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let A, B, and C be sets, and assume that $A \subseteq B$ and $B \subseteq C$. Suppose that $x \in A$. Since $x \in A$, and $A \subseteq B$, then, by Definition 4.1.4, $x \in B$. Similarly, since $x \in B$ and $B \subseteq C$, by Definition 4.1.4, $x \in C$. Then every element of A is also an element of C, hence $A \subseteq C$, as desired. $\square$.

Proposition 4.1.9: For all sets A, $\emptyset \subseteq A$, and $A \subseteq A$.

6. Prove Proposition 4.1.9 [Hint: for the first part, what elements do you need to check?]

Note: A conditional statement whose hypothesis is **never** satisfied is said to be **vacuously true**. That is, is must be true, because, logically speaking, any statement of the form $P \Rightarrow Q$ is true whenever P is false.

Interval Notation: If $a < b$ are real numbers, then:

- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$

7. For each of the following, determine whether the given statement is True or False.

(a) $2 \in \{1, 2, \{0, 1\}\}$

(b) $0 \in \{1, 2, \{0, 1\}\}$

(c) $\{0, 1\} \in \{0, 1, 2, \{0, 1\}\}$

(d) $2 \subseteq \{1, 2, \{0, 1\}\}$

(e) $\{2\} \subseteq \{1, 2, \{0, 1\}\}$

(f) $\{0, 1\} \subseteq \{1, 2, \{0, 1\}\}$

(g) $\emptyset \subseteq \{0, 1, 2, \{0, 1\}\}$

(h) $\{\emptyset\} \subseteq \{0, 1, 2, \{0, 1\}\}$

(i) $\emptyset \in \{0, 1, 2, \{0, 1\}\}$

(j) $(3, 5) \subseteq [3, 5]$

(k) $[3, 5] \subseteq (3, 5)$

(l) $[3, 5] \subseteq (3, \infty)$

(m) $[1, 3) \subseteq (-\infty, 3]$