Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 13 **Name:**______________________________
Due: At the end of class on Tuesday, October 16th

Operations on Sets: Sets are one of the most important and and fundamental mathematical objects. The goal of this chapter of content is to help familiarize you with sets and their properties. You have probably already encountered sets sometime in your previous mathematical experience. For now, we will consider "sets" as a fundamental concept to be an undefined term. We think of them as a "collection of objects". In order for a set to be *well-defined*, membership in the set must be decidable. That is, there should be a clear process or criterion that we can use to determine whether or not any *element* taken from the underlying universe is in the set (or is not in the set).

Definition 4.2.1:

- Given two sets A and B, the *union* of these sets, denoted $A \cup B$, is the set containing the elements that are either in A or in B (or in both). That is, $A \cup B = \{x : x \in A \vee x \in B\}$.

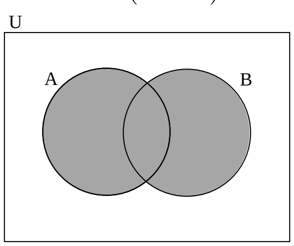
- Given two sets A and B, the *intersection* of these sets, denoted $A \cap B$, is the set containing the elements that are in both A and B. That is, $A \cap B = \{x : x \in A \wedge x \in B\}$.

- Given two sets A and B, the *difference* of these sets, denoted $B - A$, is the set containing the elements that are in B but not in A. That is, $B - A = \{x : x \in B \wedge x \notin A\}$. Note: In general, $A - B \neq B - A$. This is also sometimes denoted by B/A

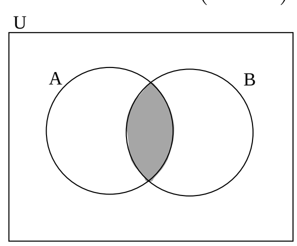
- Given a universal set U and a set A, the *complement* of the set A with respect to the set U, denoted $\overline{A}$, is the set $U - A$. That is, $\overline{A} = \{x \in U : x \notin A\}$.

Venn Diagrams for Set Operations:

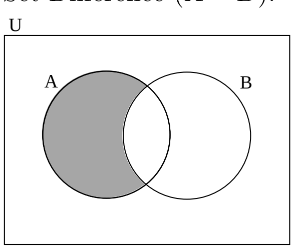
Set Union ($A \cup B$):



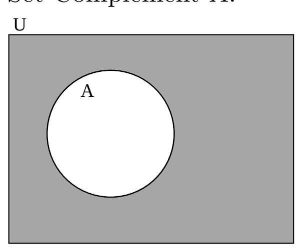
Set Intersection ($A \cap B$):



Set Difference ($A - B$):



Set Complement $\overline{A}$:



Examples: Let $A = \{a, b, c, d, e, f\}$, $B = \{a, c, e, f, g, h, i\}$, $C = \{b, c, d, f, g, h\}$ and $U = \{a, b, c, d, e, f, g, h, i, j\}$

1. $A \cup B = \{a, b, c, d, e, f, g, h, i\}$
2. $A \cap B = \{a, c, e, f\}$
3. $A - C = \{a, e\}$
4. $\overline{C} = \{a, e, i, j\}$
5. $A - (B \cup C) = A - (\{a, b, c, d, e, f, g, h, i\}) = \emptyset$
6. $A \cup (B \cap C) = A \cup (\{c, f, g, h\}) = \{a, b, c, d, e, f, g, h\}$

Definition 4.2.3 Two sets A and B are **disjoint** if their intersection is empty. That is, if $A \cap B = \emptyset$.

1. Let $A = (2, 5]$ and $B = [4, 7)$. Find:

   (a) $A \cup B$

   (b) $A \cap B$

   (c) $B - A$

   (d) $A - B$

   (e) $\overline{A}$

   (f) $\overline{B}$

Proposition 4.2.5: Let A, B, and C be sets. If $A \cap B \subseteq C$ and $x \in A - C$, then $x \notin B$.

Proof: Let A, B, and C be arbitrary sets satisfying $A \cap B \subseteq C$ and suppose $x \in A - C$. In order to obtain a contradiction, suppose that $x \in B$. Since $x \in A - C$, then, by definition of set difference, $x \in A$ and $x \notin C$. Recalling that we assumed that $x \in B$, since $a \in A$, then $x \in A \cap B$ by definition of set intersection. However, $A \cap B \subseteq C$, so $x \in A/capB$ implies that $x \in C$. But $x \notin C$. Having reached a contradiction, we conclude that, contrary to our initial assumption, $x \notin B$. $\square$.

The following theorem compiles several useful and important facts about sets and set operations. We will not take the time to prove each part, but we will look at a few – the others can be proven using similar techniques.

Theorem 4.2.6: Let A, B, and C be subsets of some universal set $\mathcal{U}$. Then:

- $A \cup A = A$
- $A \cap A = A$
- $A \cup \emptyset = A$
- $A \cap \emptyset = \emptyset$
- $A \cap B \subseteq A$
- $A \subseteq A \cup B$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup \overline{A} = \mathcal{U}$
- $A \cap \overline{A} = \emptyset$
- $\overline{\overline{A}} = A$

2. Prove that $\overline{\overline{A}} = A$

Example Proof: We will prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

"$\subseteq$" Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. We will consider these two cases separately.

Case 1: Suppose $x \in A$. Then, by definition of set union, $x \in A \cup B$. Similarly, $x \in A \cup C$. Thus. by definition of set intersection, we must have $x \in (A \cup B) \cap (A \cup C)$.

Case 2: Suppose $x \in B \cap C$. Then, by definition of set intersection, $x \in B$ and $x \in C$. Since $x \in B$, then again by the definition of set union, $x \in A \cup B$. Similarly, since $x \in C$, then $x \in A \cup C$. Hence, again by definition of set intersection, $x \in (A \cup B) \cap (A \cup C)$.

Since these are the only possible cases, then $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

"$\supseteq$" Let $x \in (A \cup B) \cap (A \cup C)$. Then, by definition of set intersection, $x \in A \cup B$ and $x \in A \cup C$. We will once again split into cases.

Case 1: Suppose $x \in A$. Then, by definition of set union, $x \in A \cup (B \cap C)$.

Case 2: Suppose $x \notin A$. Since $x \in A \cup B$, then, by definition of set union, we must have $x \in B$. Similarly, since $x \in A \cup C$, we must have $x \in C$. Therefore, by definition of set intersection, we have $x \in B \cap C$. Hence, again by definition of set union, $A \cup (B \cap C)$.

Since these are the only possible cases, then $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$

Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. $\square$

3. Let A, B, and C be sets. Show that $A \cap (B - C) = (A \cap B) - C$

Definition 4.2.7: Let A and B be sets. The **Cartesian product** of A and B is the set $A \times B = \{(x, y) \mid a \in A \,\text{and}\, y \in B\}$, that is, ordered pairs (x, y) where the first component is form the set A and the second component is from the set B. More generally, $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in A_i \,\text{for}\, i = 1, 2, \cdots, n\}$

Note: We $A^n = A \times A \times \cdots \times A$ (the Cartesian product of A with itself n times). Also, $|A|$ denotes the **cardinality** (total number of elements) in a set A.

Definition 4.2.10: Let X be a set. The **power set** of X is the set $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

Example: If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

4. Suppose $A = \{1, 2\}$ and $B = \{r, s, t\}$.

   (a) Find $A \times B$ in roster form.

   (b) Find $A \times A$ in roster form.

   (c) Find $B \times A$ in roster form.

   (d) Find $|A|$ and $|B|$.

   (e) Find $|A^4|$ and $|B^3|$.

   (f) Find $\mathcal{P}(B)$ in roster form.

Theorem 4.2.9 Let A B, C, and D be sets. Then:

- $A \times \emptyset = \emptyset = \emptyset \times A$.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
- $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. Equality does not hold in general (see the example in your textbook).

Proposition 4.2.12: Let A and B be sets. Then $A \subseteq B \Leftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$.

5. **Presentation Problems:** Let A, B, and C be arbitrary sets.

   (a) Prove Proposition 4.2.12

   (b) Prove $A - (A - B) = A \cap B$

   (c) Prove $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

   (d) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$