Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 14 **Name:**________________________________
Due: At the end of class on Tuesday, October 16th

Arbitrary Unions and Intersections:

1. Consider arbitrary sets A, B, and C.

   (a) In your own words, explain why $A \cup (B \cup C) = (A \cup B) \cup C$ and why $A \cap (B \cap C) = (A \cap B) \cap C$

   Note that as a consequence of the two exercises above, we can write $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$. Similarly, we write $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$. The next definition extends this idea to any finite number of intersections or unions.

   Definition 4.3.1: Let $n \in \mathbb{Z}^+$ and let $A_1, A_2, \cdots, A_n$ be sets. Then

   $$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x \in \mathcal{U} \mid \text{ there exists } i \in \mathbb{Z}^+ \text{ with } 1 \leq i \leq n \text{ such that } x \in A_i\}.$$

   $$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x \in \mathcal{U} \mid \text{ for all } i \in \mathbb{Z}^+ \text{ with } 1 \leq i \leq n,\ x \in A_i\}.$$

2. For each of the following collections of subsets of $\mathbb{R}$, find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$ for the given n-value.

   (a) $A_i = \{0, 1, 2, 3, \cdots i\}$ given that $n = 5$

   (b) $A_i = [0, i]$ given that $n = 5$

(c) $A_i = \{0, 1, 2, 3, \cdots i\}$ given that $n = N$

(d) $A_i = [0, i]$ given that $n = N$

Theorem 4.3.2 Let $n \in \mathbb{Z}^+$. then for all sets $A, B_1, B_2, \cdots, B_n$,

- $A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$.
- $A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$.
- $\overline{B_1 \cup B_2 \cup \cdots \cup B_n} = \overline{B_1} \cap \overline{B_2} \cap \cdots \cap \overline{B_n}$.
- $\overline{B_1 \cap B_2 \cap \cdots \cap B_n} = \overline{B_1} \cup \overline{B_2} \cup \cdots \cup \overline{B_n}$.

Proof: We will prove the first part (using proof by induction). The second part is proved in your textbook. The remaining parts are presentation problems.

Base Case: For $n = 1$, we see that $A \cup B_1 = A \cup B_1$, so the statement is true when $n = 1$.

Inductive Step: Suppose that the statement holds for all collections of sets of size m for some $m \geq 1$. That is, suppose that for any collection of sets $C, D_1, D_2, \cdots D_m$, we have $C \cup (D_1 \cap D_2 \cap \cdots \cap D_n) = (C \cup D_1) \cap (C \cup D_2) \cap \cdots \cap (C \cup D_n)$.

Given a collection of sets $A, B_1, B_2, \cdots, B_m, B_{m+1}$, consider $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m \cap B_{m+1})$. Notice that $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m \cap B_{m+1}) = A \cup ((B_1 \cap B_2 \cap \cdots \cap B_m) \cap B_{m+1}) = A \cup (B_1 \cap B_2 \cap \cdots \cap B_m) \cap (A \cup B_{m+1})$ [by Theorem 4.2.6].

By the induction hypothesis, with $C = A$ and $D_i = B_i$, we have $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$. Hence $A \cup (B_1 \cap B_2 \cap \cdots \cap B_m \cap B_{m+1}) = ((A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)) \cap (A \cup B_{m+1})$, which verifies the $m+1$ case.

Therefore, $A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$. $\square$.

The next definition extends thus idea to an infinite collection of sets (a collection indexed – that is, put in 1-1 correspondence through labeling – by the positive integers.

Definition 4.3.3: Given sets A_i, $i \in \mathbb{Z}^+$, with underlying universal set $\mathcal{U}$,

$$\bigcup_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \text{ there exists } i \in \mathbb{Z}^+ \text{ such that } x \in A_i\}.$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \text{ for all } i \in \mathbb{Z}^+,\ x \in A_i\}.$$

3. For each of the following indexed collections of subsets of $\mathbb{R}$, A_i for $i \in \mathbb{Z}^+$, find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

(a) $A_i = \{0, 1, 2, 3, \cdots i\}$

(b) $A_i = [0, i)$

(c) $A_i = [0, \frac{1}{i}]$

Definition 4.3.6: Let I be a non-empty set and let Given sets $\{A_i, \mid i \in I\}$ be an indexed family of sets, relative to some universal set $\mathcal{U}$. Then

- For each $j \in I$, $\bigcap_{i=I} A_i \subseteq A_j$.
- For each $j \in I$, $A_j \subseteq \bigcup_{i=I} A_i$.
- $B \cup \bigcap_{i=I} A_i = \bigcap_{i=I} (B \cup A_i)$
- $B \cap \bigcup_{i=I} A_i = \bigcup_{i=I} (B \cap A_i)$
- $\overline{\bigcup_{i=I} A_i} = \bigcap_{i=I} \overline{A_i}$
- $\overline{\bigcap_{i=I} A_i} = \bigcup_{i=I} \overline{A_i}$

4. Explain, in your own words, why the first two parts of the Theorem above make sense.

5. **Presentation Problems:** Let A, B, and C be arbitrary sets.

(a) Prove Proposition 4.2.12

(b) Prove $A - (A - B) = A \cap B$

(c) Prove $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(d) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(e) Prove or Disprove: If $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.

(f) Prove or Disprove: If $A \subseteq B \cap C$ then $A \subseteq B$ and $A \subseteq C$.

(g) Prove or Disprove: If $A - C \subseteq B - C$ then $A \subseteq B$.

(h) Prove or Disprove: $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

(i) Prove or Disprove: $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(j) Express the set $\{1\}$ as the intersection of a collection of distinct, non-empty intervals in $\mathbb{R}$ indexed by $\mathbb{Z}^+$.