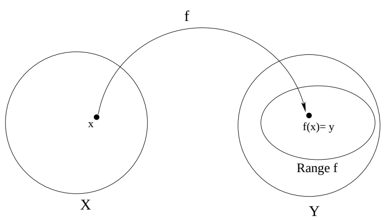
Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 15 **Name:**________________________________
Due: At the end of class on Tuesday, October 23rd

Functions:

Definition 5.1.1: Let X and Y be non-empty sets. A **function** f from the set X to the set Y, denoted by $f : X \to Y$ or by $f(x)$ is a correspondence that assigns to each element $x \in X$ a unique element $y \in Y$.



We say that f **maps** x to y (written $y = f(x)$) when y is the unique element of Y that is assigned by f to an element x in X. We call y the **image of** x **under** f, and we call x a **preimage of** y **under** f. We say that the function f maps the set X into the set Y.

Continuing to use the notation defined above, we say that the set X is the **domain** of the function f and the set Y is the **codomain** of f.

Definition 5.1.4 Let X and Y be sets and let $f : X \to Y$. The **range** of f (also called the image of X under f) is the set $\{y \in Y \,|\, (\exists x \in X[y = f(x)]\} = \{f(x) \,|\, x \in X\}$. We will often denote this as either $ran\, f$ or $im\, f$.

Definition 5.1.5 Let X and Y be sets and let $f : X \to Y$. The **graph of** f is the set $G_f = \{(x, y) \in X \times Y \,|\, y = f(x)\} = \{(x, f(x)) \,|\, x \in X\}$.

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by the rule $f(n) = 0$ if n is even and $f(n) = 1$ if n is odd.
   (a) Explain, in your own words, why f, as defined above is a function.

   (b) Find the following:
      i. $f(12)$
      ii. $f(27)$
      iii. the image of 0 under f.
      iv. The preimage of 0 under f
      v. The preimage of 1 under f
      vi. The preimage of 2 under f.
      vii. The domain of f
      viii. The range of f

2. Let $f(x) = \sin x$ and $g(x) = \cos\left(x - \frac{\pi}{2}\right)$ be functions from $\mathbb{R} \to \mathbb{R}$.

   (a) Find the domain and range of $f(x)$.

   (b) Find the domain and range of $g(x)$.

   (c) Draw the graph of $f(x)$.

   (d) Draw the graph of $g(x)$.

Definition 5.1.7: Let A, B, C, and D be sets. Let $f : A \to B$ and $g : C \to D$ be functions. Then $f = g$ if:

- $A = C$ and $B = D$
- For all $x \in A$, $f(x) = g(x)$.

Intuitively speaking, this definition tells us that a function is determined by its underlying correspondence, not its specific formula or rule. Another way to think of this is that a function is determined by the set of points that occur on its graph.

3. Given a specific example of two functions that are defined by different rules (or formulas) but that are equal as functions.

4. Consider the functions $f(x) = x$ and $g(x) = \sqrt{x^2}$. Find:

(a) A domain for which these functions are equal.

(b) A domain for which these functions are **not** equal.

Definition 5.1.9 Let X be a set. The **identity function on** X is the function $I_X : X \to X$ defined by, for all $x \in X$, $I_X(x) = x$.

5. Let $f(x) = x\cos(2\pi x)$. Prove that $f(x)$ is the identity function when $X = \mathbb{Z}$ but not when $X = \mathbb{R}$.

Definition 5.1.10 Let $n \in \mathbb{Z}$ with $n \geq 0$, and let $a_0, a_1, \cdots, a_n \in \mathbb{R}$ such that $a_n \neq 0$. The function $p : \mathbb{R} \to \mathbb{R}$ is a **polynomial of degree** n **with real coefficients** $a_0, a_1, \cdots, a_n$ if for all $x \in \mathbb{R}$, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

The **zero polynomial** is the function $q : \mathbb{R} \to \mathbb{R}$ such that $q(x) = 0$ for all $x \in \mathbb{R}$. The degree of this polynomial is undefined.

6. True or False: $p(x) = 1$ is a polynomial (briefly justify your answer).

7. True or False: $p(x) = 1$ is the identity function on $\mathbb{R}$ (briefly justify your answer).

8. True or False: Every polynomial has domain $\mathbb{R}$ (briefly justify your answer).

9. True or False: Every polynomial has image $\mathbb{R}$ (briefly justify your answer).

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{x+4}{x^2-9}$. For what values of x is $f(x)$ defined?
Note: we often call this the *implicit* or *natural domain* of f.

11. Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $g(m, n) = mn$.

(a) Find the image of the function g.

(b) Find the preimage of 0 under g.