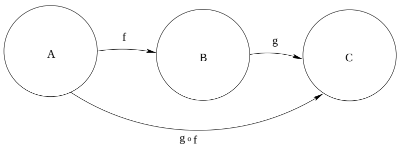
Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 16 **Name:**______________________________
Due: At the end of class on Thursday, October 25th

**Function Composition:**

**Definition 5.2.1:** Let $A$, $B$, $C$, and $D$ be sets. Let $f : A \to B$ and $g : C \to D$ with $ran\, f \subset C$. The **composite (composition) of** $f$ **and** $g$ is the function $g \circ f : A \to D$ defined by, for all $x \in A$, $(g \circ f)(x) = g(f(x))$.

**Note:** When $f : A \to B$ and $g : B \to C$, then we the following diagram illustrates the composite function $g \circ f : A \to C$.




1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x - 1$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

   (a) Find $(g \circ f)(4)$

   (b) Find $(f \circ g)(4)$

   (c) Find $(g \circ f)(x)$

   (d) Find $(g \circ f)(x)$

   (e) Based on this example, what do you conclude about how $(g \circ f)(x)$ and $(f \circ g)(x)$ compare in general?

2. Let $f(x) = \begin{cases} x-2 & \text{if } x \geq 1 \\ x+2 & \text{if } x < 1 \end{cases}$ and let $g(x) = \begin{cases} 2x & \text{if } x > 4 \\ \frac{1}{2}x & \text{if } x \leq 4 \end{cases}$

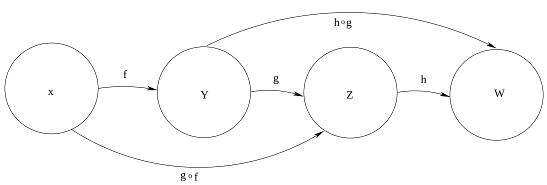
(a) Find $(g \circ f)(1)$

(b) Find $(f \circ g)(4)$

(c) Find $(g \circ f)(x)$

**Proposition 5.2.5:** Let $X$, $Y$, $Z$, and $W$ be sets. Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$. Then:

- $(h \circ g) \circ f = h \circ (g \circ f)$; that is, function composition is associative.
- $f \circ I_X = f = I_Y \circ f$.




**Proof:** Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$. To prove the first part, we must demonstrate that $(h \circ g) \circ f = h \circ (g \circ f)$. Note that both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ define functions from $X$ into $W$. Let $x \in X$. Suppose $y = f(x)$. Then $(h \circ g) \circ f)(x) = (h \circ g)(f(x)) = (h \circ g)(y) = h(g(y)) = h(g(f(x)))$. Similarly, $h \circ (g \circ f)(x) = h((g \circ f)(x))$. But $(g \circ f)(x) = g(f(x)) = g(y)$, so $h((g \circ f)(x)) = h(g(y)) = h(g(f(x)))$. Hence, for any $x \in X$, $(h \circ g) \circ f)(x) = h \circ (g \circ f)(x)$. Hence $(h \circ g) \circ f = h \circ (g \circ f)$.

Let $x \in X$. Recall that by definition, for any $x \in X$, $I_X(x) = x$. Therefore, $(f \circ I_X)(x) = f(I_X(x)) = f(x)$. Hence $f \circ I_X = f$. The proof that $f = I_Y \circ f$ is a similar proof that is left to you. □.

3. Prove that $f = I_Y \circ f$.

4. Find a distinct pair of functions, $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = g \circ f = I_{\mathbb{R}}$.

**Definition 5.3.1** Let $X$ and $Y$ be sets and let $f : X \to Y$.

- The function $f$ is **one-to-one** if $(\forall x_1, x_2 \in X)[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$, or, equivalently, $(\forall x_1, x_2 \in X)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$.

  When $f$ is one-to-one, we say that $f$ is **injective**, or that $f$ is **an injection**.

- The function $f$ is **onto** if $(\forall y \in Y)[\exists x \in X \mid y = f(x)]$.
  When $f$ is onto, we say that $f$ is **surjective**, or that $f$ is **a surjection**. Note that $f : X \to Y$ is onto iff $ran\, f = Y$.
- The function $f$ is **bijective**, or is a bijection (or a 1-1 **correspondence**), if $f$ is both an injection and a surjection. That is, $f$ is both 1-1 and onto.

**Example:** Let $f(x) = 3x - 2$. We will prove that $f(x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

First, to see that $f(x)$ is 1-1, suppose that $f(a) = f(b)$. Then $3a - 2 = 3b - 2$. Adding 2 to both sides gives $3a = 3b$. Dividing both sides by 3 gives $a = b$. Hence $f$ is 1-1.

To see that $f$ is onto, let $y \in \mathbb{R}$. Let $x = \frac{y+2}{3}$ (note that $x \in \mathbb{R}$). Then $f(x) = f\left(\frac{y+2}{3}\right) = 3 \cdot \left(\frac{y+2}{3}\right) - 2 = (y + 2) - 2 = y$. This shows that $y \in im\, f$. Since $y$ was arbitrary, this demonstrates that $f$ is onto.

Since $f$ is both 1-1 and onto, then $f$ is a bijection. $\square$.

**Theorem 5.3.6** Let $f : X \to Y$ and let $R = ran\, f$ be the range of $f$. Then the function $g : X \to R$ defined by $g(x) = f(x)$ for all $x \in X$ is onto.

5. For each of the following functions, determine (with proof) whether or not the function is 1-1, onto, both, or neither. For those that are not onto, find the range of the function.

   (a) $f(x) = x^2$ where $f : \mathbb{R} \to \mathbb{R}$.

(b) $g(n) = 2n$ where $g : \mathbb{N} \to \mathbb{N}$.

(c) $h(x) = 2x$ where $h : \mathbb{R} \to \mathbb{R}$.

(d) $p(x, y) = x + y$ where $p : \mathbb{R}^2 \to \mathbb{R}$.