

Due: At the end of class on Tuesday, October 30th

**Invertible Functions:**

**Theorem 5.3.6:** Let  $f : X \rightarrow Y$ , and let  $\text{ran } f$  be the range of  $f$ . Then the function  $g : X \rightarrow \text{ran } f$  defined by  $g(x) = f(x)$  for all  $x \in X$  is onto.

1. Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \sqrt{x} - 4$ . Find the image of the function  $f$ , and explain why the associated function  $g : \mathbb{R} \rightarrow \text{im } f$  given by  $g(x) = f(x)$  for all  $x \in X$  is onto.

2. Prove Theorem 5.3.6

**Theorem 5.3.8:** Let  $f : X \rightarrow Y$  be onto. Then there exists a subset  $X_0 \subseteq X$  such that restricting the domain of  $f$  to  $X_0$  yields a 1-1 and onto function; i.e., the function  $g : X_0 \rightarrow Y$  defined by  $g(x) = f(x)$  for all  $x \in X_0$  is a bijection.

**Notes:**

- The proof of Theorem 5.3.8 requires the Axiom of Choice (a controversial axiom which you can read about in section 4.4 of your textbook, or other places). The main requirement is that for each  $y \in Y$ , one must be able to select a specific  $x \in X$  such that  $f(x) = y$ . This is straightforward when  $Y$  is a finite set, but can be problematic when  $Y$  is infinite.
- In most of the examples we have looked at so far, we have shown a function is onto by using algebra to produce an element  $x$  in the domain such that  $f(x) = y$ . In practice, using algebra to find such  $x$ -values precisely can be challenging. Results like the Intermediate Value Theorem are helpful as they allow us to prove that an input leading to a given value exists without finding its exact value.
- Results like these are said to **non-constructive** existential proofs (in contrast to what are called **constructive** existential proofs) since they show something exists, but do not actually find the object under consideration. Constructive vs. Non-Constructive proof is another area of controversy in mathematics.

3. Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \sin x$ . Find  $\text{im } f$ . Then find a set  $X_0$  such that the function  $g : X_0 \rightarrow \text{im } f$  defined by  $g(x) = f(x)$  for all  $x \in X_0$  is a bijection.

**Theorem 5.3.10** Let  $X$ ,  $Y$ , and  $Z$  be sets. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be functions.

- If  $f$  and  $g$  are both 1-1, then  $g \circ f$  is 1-1.
- If  $f$  and  $g$  are both onto, then  $g \circ f$  is onto.
- If  $f$  and  $g$  are both bijections, then  $g \circ f$  is a bijection.
- If  $g \circ f$  is 1-1, then  $f$  is 1-1, but  $g$  need not be.
- If  $g \circ f$  is onto, then  $g$  is onto, but  $f$  need not be.

**Proof:** If  $f$  and  $g$  are both 1-1, then  $g \circ f$  is 1-1.

Suppose  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are 1-1 functions. Let  $a, b \in X$  and suppose  $(g \circ f)(a) = (g \circ f)(b)$ . Then  $g(f(a)) = g(f(b))$ . Since  $g$  is 1-1, we must have  $f(a) = f(b)$ . Then, since  $f$  is 1-1, we must have  $a = b$ . Hence  $g \circ f$  is 1-1.  $\square$ .

4. Prove that If  $f$  and  $g$  are both onto, then  $g \circ f$  is onto.

5. Give an example of functions  $f$  and  $g$  such that  $g \circ f$  is 1-1 and  $f$  is 1-1, but  $g$  is not 1-1.

**Definition 5.4.1** Let  $X$  and  $Y$  be sets and let  $f : X \rightarrow Y$ . We say that  $f$  is **invertible** if there exists a function  $g : Y \rightarrow X$  such that for all  $x \in X$  and for all  $y \in Y$ ,  $y = f(x)$  if and only if  $x = g(y)$ .

When this definition is satisfied, we say that the function  $g$  is an **inverse function** of  $f$ .

**Proposition 5.4.3** Let  $X$  and  $Y$  be sets, and let  $f : X \rightarrow Y$  and let  $g : Y \rightarrow X$ . Then  $f$  is invertible and  $g$  is an inverse function of  $f$  if and only if  $g \circ f = I_X$  and  $f \circ g = I_Y$ .

**Proof:** Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ .

“ $\Rightarrow$ ”. Assume that  $f$  is invertible and that  $g$  is an inverse function of  $f$ . Then, by definition,  $y = f(x) \Leftrightarrow x = g(y)$ . Consider  $g \circ f$ . Notice that  $g \circ f : X \rightarrow X$ . Let  $x \in X$  and suppose  $y = f(x)$ . Then  $(g \circ f)(x) = g(f(x)) = g(y) = x = I_X(x)$ . Hence  $g \circ f = I_X$ .

Similarly,  $f \circ g : Y \rightarrow Y$ . Let  $y \in Y$  and suppose  $x = g(y)$ . Then  $(f \circ g)(y) = f(g(y)) = f(x) = y = I_Y(y)$ . Hence  $f \circ g = I_Y$ .

“ $\Leftarrow$ ”. Assume that  $f$  and  $g$  satisfy  $g \circ f = I_X$  and  $f \circ g = I_Y$ .

Let  $x \in X$  and suppose  $y = f(x)$ . Then  $g(y) = g(f(x)) = (g \circ f)(x) = I_X(x) = x$ . Hence  $g(y) = x$ .

Similarly, let  $y \in Y$  and suppose  $x = g(y)$ . Then  $f(x) = f(g(y)) = (f \circ g)(y) = I_Y(y) = y$ . Hence  $f(x) = y$ .

Thus  $f$  is invertible and  $g$  is an inverse function of  $f$ .  $\square$ .

**Corollary 5.4.4:** Let  $X$  and  $Y$  be sets and let  $f : X \rightarrow Y$ . Then  $f$  is invertible if and only if there is a function  $g : Y \rightarrow X$  such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ .

6. Let  $f(x) = 2x - 3$  and  $g(x) = \frac{x+3}{2}$  be functions from  $\mathbb{R} \rightarrow \mathbb{R}$ . Prove that  $f$  and  $g$  are inverse functions.

**Theorem 5.4.7** Let  $X, Y$  be sets and let  $f : X \rightarrow Y$ .

- Then  $f$  is invertible if and only if  $f$  is a bijection.
- If  $f$  is invertible, then its inverse function is unique.

**Proof:** See p. 117 in your textbook.

**Notation 5.4.8:** When  $f : X \rightarrow Y$  is invertible, the unique inverse function is denoted by  $f^{-1}$ , and  $f^{-1} : Y \rightarrow X$ .

**Corollary 5.4.9** Let  $X$  and  $Y$  be sets and let  $f : X \rightarrow Y$ . If  $f$  is a bijection, then  $f^{-1} : Y \rightarrow X$  is a bijection.

**Corollary 5.4.10** Let  $X$  and  $Y$  be sets and assume  $f : X \rightarrow Y$  is 1-1. Then the function  $g : X \rightarrow \text{ran } f$  defined by, for all  $x \in X$ ,  $g(x) = f(x)$  is invertible.

**Corollary 5.4.11** Let  $X$  and  $Y$  be sets and let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ . If  $g \circ f = I_X$  and  $f \circ g = I_Y$ , then  $g = f^{-1}$  and  $f = g^{-1}$ .

7. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by  $f((x, y)) = (2y, 3x)$ . Determine whether or not  $f$  is invertible. If it is invertible, find  $f^{-1}$ , its inverse function.

**Definition 5.5.1** Let  $f : X \rightarrow Y$  and  $A \subseteq X$ . The **image of  $A$  under  $f$**  is the set  $\{y \in Y \mid (\exists x \in A)[y = f(x)]\} = \{f(x) \mid x \in A\}$ . We sometimes denote the image set as  $f[A]$ .

**Definition 5.5.2** Let  $f : X \rightarrow Y$  and  $B \subseteq Y$ . The **inverse image of  $B$  under  $f$**  (also called the **preimage** of  $B$  under  $f$ ) is the set  $\{x \in X \mid f(x) \in B\}$ . We sometimes denote the preimage of a set as  $f^{-1}[B]$  (even in situations where  $f$  is not invertible).

8. Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be given by  $f(k) = k + 11$ . Let  $E$  be the set of even integers and let  $F$  be the set of odd integers. Let  $A = \{k \in \mathbb{Z} \mid k \geq 0\}$ . and let  $B = \{k \in \mathbb{Z} \mid k < 0\}$

(a) Find  $f[E]$ .

(b) Find  $f[A]$ .

(c) Find  $f^{-1}[F]$

(d) Find  $f^{-1}[B]$