Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 21 **Name:**______________________________
Due: At the end of class on Thursday, November 15th

Equivalence Relations:

Definition 7.1.1: Let A and B be sets. A **binary relation** $\mathcal{R}$ from A to B is a subset of $A \times B$. A **binary relation** on a set A is a subset of $A \times A$.

Example 1: Let $A = \{L, M, N, O\}$ and let $B = \{2, 3, 5, 7, 11\}$. Let $\mathcal{R} = \{(L, 2), (M, 3), (M, 5), (M, 7), (M, 11), (N, 2), (O, 5), (O, 11)\}$. Notice that $\mathcal{R} \subseteq A \times B$, so $\mathcal{R}$ does define a relation from A to B. We see that M is "$\mathcal{R}$-related" to 3, 5, 7, and 11, since the ordered pairs $(M, 3)$, $(M, 5)$, $(M, 7)$ and $(M, 11)$ are all in $\mathcal{R}$. However, M is **not** "$\mathcal{R}$-related" to 2, since the ordered pair $(M, 2)$ is not an element of $\mathcal{R}$.

Example 2: Let $A = \mathbb{R}$ and let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by $F(x) = 3x - 2$ for all $x \in \mathbb{R}$. Then F can be thought of as a relation on $\mathbb{R}$ as it defines a subset of $\mathbb{R} \times \mathbb{R}$. For example, since $F(0) = -2$ and $F(3) = 7$, then the ordered pairs $(0, -2)$ and $(3, 7)$ are elements of the relation defined by F, while $(2, 3)$ is **not** an element of the relation defined by F.

Notation: When $\mathcal{R} \subseteq A \times B$ is a relation from A to B, we say that the elements $a \in A$ and $b \in B$ are $\mathcal{R}$-related when $(a, b) \in \mathcal{R}$. We often use the following notation to express these relationships.

Definition 7.1.3: Let A and B be sets, and let $\mathcal{R} \subseteq A \times B$ be a relation from A to B. For $a \in A$ and $b \in B$, we write

- $a\mathcal{R}b$ if and only if $(a, b) \in \mathcal{R}$ and
- $a \not\mathcal{R} b$ if and only if $(a, b) \notin \mathcal{R}$

Examples:

- Applying this notation to Example 1 above, we would write $M\mathcal{R}3$ and $N\mathcal{R}2$, while $M\not\mathcal{R}2$.
- Similarly, for the relation F in Example 2, we have $2\mathcal{R}4$ and $3\mathcal{R}7$, while $2\not\mathcal{R}1$

1. In your own words, explain why any function $f : A \to B$ defines a relation from A to B.

2. Define a relation between $\mathbb{Z}$ and $\mathbb{Z}$. List three pairs that are related under your relation and two pairs that are not related.

3. Give a specific example of a relation that is **not** a function. Be sure to provide specific evidence that shows your relation is not a function.

More Notation: A common way that relations between sets are represented is with the symbol "$\sim$". Here is a nice example from your book that illustrates this notation.

Example 3: Let $\mathbb{Z}^* - \mathbb{Z} - \{0\}$. We define the relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}^*$ as follows:

For all $a, c, \in \mathbb{Z}$ and all $b, d \in \mathbb{Z}^*$, $(a, b) \sim (c, d)$ if and only if $ad = bc$. Note that the elements of $\sim$ are actually ordered pairs of ordered pairs. That is, $\sim$ consists of elements of the form $((a, b), (c, d))$. For example, $(2, 4) \sim (3, 6)$ since $(2) \cdot (6) = (4) \cdot (3)$.

4. Find five distinct pairs of ordered pairs that are related under $\sim$.

5. Find three pairs of ordered pairs that are **not** related under $\sim$.

6. Describe any patterns you see regarding which pairs are $\sim$ related and which pairs are not $\sim$ related.

Definition 7.1.7: Let $\mathcal{R}$ be a relation from a set X to a set Y. The **domain** of $\mathcal{R}$ is the set $\operatorname{dom} \mathcal{R} = \{a \in X \mid (\exists b \in Y)[(a, b) \in \mathcal{R}\}$. Note that we are simply adapting our definition of the domain of a function to this new context.

Definition 7.1.8: A relation $\mathcal{R}$from a set X to a set Y is a **function** if for all $a \in \operatorname{dom} \mathcal{R}$, there exists a unique $b \in Y$ such that $(a, b) \in \mathcal{R}$. That is, $(\forall a \in \operatorname{dom} \mathcal{R})(\exists! b \in Y)[(a, b) \in \mathcal{R}]$. Furthermore, if $\operatorname{dom} \mathcal{R} = X$, then $\mathcal{R} : X \to Y$.

Definition 7.2.1: Let $\sim$ be a relation on a set A.

- We say that $\sim$ is **reflexive** if for all $a \in A$, $a \sim a$.
- We say that $\sim$ is **symmetric** if for all $a, b \in A$, if $a \sim b$ then $b \sim a$.
- We day that $\sim$ is **transitive** if for all $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$.
- The relation $\sim$ is called an **equivalence relation** on A if $\sim$ is reflexive, symmetric, and transitive.

7. Consider the relation "$\geq$" defined on $\mathbb{R}$ via $(a, b) \in \mathcal{R}$ if and only if $a \geq b$.

   (a) Determine whether or not "$\geq$" is reflexive.

   (b) Determine whether or not "$\geq$" is symmetric.

   (c) Determine whether or not "$\geq$" is transitive.

   (d) Determine whether or not "$\geq$" is an equivalence relation.

Theorem 7.2.2: Let n be any natural number. Then congruence modulo n is an equivalence relation on $\mathbb{Z}$. In other words, the relation $\sim$ defined by $a \sim b$ if and only if $a \equiv b (\text{mod} n)$ is an equivalence relation.

8. Prove Theorem 7.2.2