Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 23 **Name:**______________________________
Due: At the end of class on Tuesday, November 27th

Partially Ordered Sets:

Definition 7.4.1: Let $\sim$ be a relation on a set A.

- We say that $\sim$ is **antisymmetric** if for all $a, b \in A$, if whenever $a \sim b$ and $b \sim a$, then $a = b$.
- The relation $\sim$ is called an **partial ordering** on A if $\sim$ is reflexive, antisymmetric, and transitive.

1. Consider the following equivalence relations on the set $A = \{0, 1, 2, 3\}$:
   - $\mathcal{R}_1 = \{(0,0), (1,2), (2,3), (3,3)\}$
   - $\mathcal{R}_2 = \{(0,0), (0,1), (1,1), (1,2), (2,2), (2,3), (3,3)\}$
   - $\mathcal{R}_3 = \{(0,0), (1,1), (2,2), (3,3)\}$
   - $\mathcal{R}_4 = \{(1,2), (2,1), (1,1)\}$
   - $\mathcal{R}_5 = \{(0,0), (0,1), (1,0), (1,1), (2,2), (1,3), (3,1), (3,3)\}$

   (a) Which of these relations are reflexive?

   (b) Which of these relations are symmetric?

   (c) Which of these relations are antisymmetric?

   (d) Which of these are equivalence relations?

   (e) Which of these are partial orderings?

Example 1: The relation $\leq$ on the set $\mathbb{Z}$ is a partial ordering. We often abbreviate this as $(\mathbb{Z}, \leq)$ and we say that $(\mathbb{Z}, \leq)$ is a **partially ordered set** (or a **poset**). Note that we have already verified that this relation is both reflexive and transitive. The fact that this relation is antisymmetric follows from the fact that if $a \leq b$ and $b \leq a$, then $a = b$.

2. Consider the relation $\sim$ on the set $\mathbb{Z}^+$ defined as follows: $a \sim b$ if and only if $a \mid b$. We will denote this relation as $(\mathbb{Z}^+, \mid)$. Prove that $(\mathbb{Z}^+, \mid)$ is a poset.

Definition 7.4.2: Let $(A, \preceq)$ be a poset.

- A pair of elements $a, b \in A$ are **comparable** if either $a \preceq b$ or $b \preceq a$.
- If, for a given pair of elements $a, b \in A$, we have $a \not\preceq b$ or $b \not\preceq a$, then we say that a and b are **incomparable** in $(A, \preceq)$.
- If **every** pair of elements of A are comparable under $\preceq$, then we say that $(A, \preceq)$ is a **totally ordered set** or a **linearly ordered set**.
- If $(A, \preceq)$ is a totally ordered set such that every nonempty subset of A has a least element, then we call $(A, \preceq)$ a **well-ordered set**.

3. Show that $(\mathbb{Z}^+, \mid)$ is **not** a totally ordered set by providing an example of a pair of elements that are **incomparable**.

4. Determine whether or not $(\mathbb{Z}, \leq)$ is a totally ordered set.

5. Determine whether or not $(\mathbb{N}, \leq)$ is a well-ordered set.

Note: The Principle of Mathematical Induction can be modified to work on any well ordered set.

The remaining problems on this worksheet are all presentation problems that can be presented either today or next Tuesday when we get back from Thanksgiving Break. Note each of the four parts of these problems count as a separate presentation problem, but a single individual may only present one part of each problem.

6. For each of the following relations:

   - Determine whether or not the relation is an equivalence relation. If the relation is an equivalence relation, describe its equivalence classes.
   - Determine whether or not the relation is a poset. Either find a pair of incomparable elements or explain why this is not possible.

   (a) The relation $\sim$ on $\mathbb{R}$ defined by $a \sim b$ if and only if $|a| = |b|$.

   (b) The relation $\sim$ on the set of students as MSUM defined by $a \sim b$ if and only if person a has shaken hands with person b.

   (c) Given a set A, the relation $\sim$ on $\mathcal{P}(A)$ given by $A_1 \sim A_2$ if and only if $A_1 \subseteq A_2$.

7. Let $A = \{a, b, c, d\}$. For each of the following, find the smallest equivalence relation containing the given ordered pairs. By "find", we mean that you should give both the underlying partition **and** a complete listing of all ordered pairs in the relation.

   (a) $A_1 = \{(a, a), (a, b), (b, c), (d, d)\}$

   (b) $A_2 = \{(a, a), (b, a), (c, d)\}$

   (c) $A_3 = \{(a, b), (a, c), (a, d)\}$

   (d) $A_4 = \{(a, b)\}$