Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 24 **Name:**________________________
Due: At the end of class on Thursday, November 29th

Graphs, Diagrams, and Partially Ordered Sets:

Definition 7.5.1: A **graph** $\Gamma = (V, E)$ consists of a set V, a nonempty set of **vertices** and E, a set of **edges**. Each edge has either one or two associated vertices, called its **endpoints**. An edge is said to **connect** its endpoints. Note that the edges in these graphs are thought of as being **undirected**.
Notes:

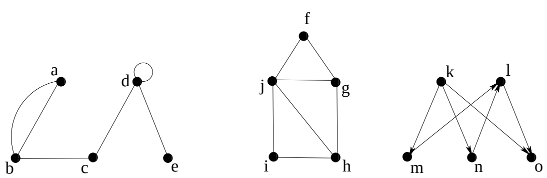
- While V, the set of vertices, is required to be nonempty, E, the set of edges could be empty.
- In many specific applications, V and E are finite sets. In this case, we call the resulting graph a **finite graph**. If the vertex set V is infinite, then we call the graph an **infinite graph**.

Definition 7.5.2: A **simple graph** is graph that has some extra restrictions placed on the type of edges that are allowed. In particular, loops and multiple edges are **not** allowed in a simple graph (see below for descriptions of these).

- As alluded to in definition 7.5.1, edges can begin and end at the same vertex. Such edges are called **loops**.
- If two (or more edges) have the same endpoints, these edges are called **multiple edges**.

Definition 7.5.3: A **directed graph** or **digraph** $\Gamma = (V, E)$ consists of a set V, a nonempty set of **vertices** and E, a set of **directed edges**. Each directed edge is associated with an ordered pair of vertices. The directed edges associated with the ordered pair (u, v) is said to **start** at u and **end** at v.

1. The following diagram illustrates examples of a graph, and simple graph, and a digraph:



   (a) Use ordered pairs to describe the elements of E_3, the directed edges in the third graph in the diagram above.

   (b) Use sets of cardinality two to describe the elements of E_2, the undirected edges in the second graph in the diagram above.

   (c) Explain why it is not possible to describe E_1 , the edges in the first graph in the diagram above, using set notation.

2. Consider the following equivalence relations on the set $A = \{0, 1, 2, 3\}$

- $\mathcal{R}_1 = \{(0,0), (1,2), (2,3), (3,3)\}$
- $\mathcal{R}_2 = \{(0,0), (0,1), (1,1), (1,2), (2,2), (2,3), (3,3)\}$
- $\mathcal{R}_3 = \{(0,0), (1,1), (2,2), (3,3)\}$
- $\mathcal{R}_4 = \{(1,2), (2,1), (1,1)\}$
- $\mathcal{R}_5 = \{(0,0), (0,1), (1,0), (1,1), (2,2), (1,3), (3,1), (3,3)\}$

(a) Create a graph representing each relation by letting $V = A$ and creating a directed edge corresponding to each ordered pair in the relation.

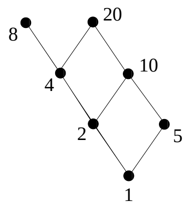
(b) Explain, in your own words, how you can tell, based on the graph, whether or not a relation is reflexive.

(c) Explain, in your own words, how you can tell, based on the graph, whether or not a relation is symmetric.

(d) Explain, in your own words, how you can tell, based on the graph, whether or not a relation is transitive.

Definition 7.5.4 A **Hasse Diagram** is a graph representation of a partial order. The vertices are the elements of the underlying set. We omit "loop" edges that indicate reflexive relationships. We draw edges connecting vertices that are comparable. However, rather than adding arrows to show the direction of the comparability, we instead vertically align the vertices so that the "smaller" vertex is below the "bigger" vertex. We also omit edges that can be inferred by applying transitivity to existing edges in the diagram (so we only connect vertices that are "directly" related).

Example: Here is the Hasse Diagram for the poset $(A, |)$, where $A = \{1, 2, 4, 5, 8, 10, 20\}$.



Definition 7.5.5 Given a poset $(A, \preceq)$:

- An element $a \in A$ is **maximal** if there is no $b \in A$ such that $a \prec b$.
- An element $a \in A$ is **minimal** if there is no $b \in A$ such that $b \prec a$.
- An element $a \in A$ is the **greatest element** if $b \preceq a$ for all $b \in A$.
- An element $a \in A$ is the **least element** if $a \preceq b$ for all $b \in A$.
- An element u is an **upper bound** of a subset $B \subseteq A$ if $b \preceq u$ for all $b \in B$.
- An element ℓ is a **lower bound** of a subset $B \subseteq A$ if $\ell \preceq b$ for all $b \in B$.
- An element x is the **least upper bound** of a subset $B \subseteq A$ if x is an upper bound and $x \preceq u$ for all other upper bounds u of the set B.
- An element y is the **greatest lower bound** of a subset $B \subseteq A$ if y is a lower bound and $\ell \preceq y$ for all other lower bounds ℓ of the set B.

3. Consider the poset $(A, |)$, where $A = \{3, 5, 9, 15, 24, 45\}$.

   (a) Draw the Hasse diagram for this poset.

   (b) Find the maximal elements.

   (c) Find the minimal elements.

   (d) Is there a greatest element?

   (e) Is there a least element?

   (f) Find all upper bounds of $\{3, 5\}$

   (g) Find the least upper bound of $\{3, 5\}$

   (h) Find all lower bounds of $\{15, 45\}$

   (i) Find the greatest lower bound of $\{15, 45\}$

4. Draw the Hasse Diagram for the poset $(\mathcal{P}(A), \subseteq)$ where $A = \{a, b, c\}$.

The remaining problems on this worksheet are all presentation problems that can be presented either today or in class on Thursday.

5. The **degree** of a vertex in a graph Γ is the number of "edge ends" that are incident with that particular vertex (note that "loop" edges contribute two to the degree of a vertex since both "ends" of the loop are incident with the vertex. Prove that the sum of the degrees of the vertices of a finite graph is equal to twice the total number of edges in the graph.

6. The **complete graph on n vertices**, denoted by K_n is a simple graph in which there is exactly one edge between each distinct pair of vertices. Find the number of edges in K_1, K_2, K_3, and K_4 and then conjecture a formula for a function whose output is the number of edges in K_n in terms of n.

7. A **lattice** is a partial order in which every pair of elements has both a least upper bound and a greatest lower bound. Notice that the poset $(A, |)$, where $A = \{3, 5, 9, 15, 24, 45\}$ is **not** a lattice. Make this poset into a lattice by adding additional elements to the set A. Try to do so using the minimum number of additional elements.