Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 2 **Name:**________________________________
Due: Tuesday, September 4th

Recall: Definition 1.1.6 Two statements (involving the same statement letters) are **logically equivalent** if they have the "same" truth table.

Proposition 1.1.7 (DeMorgan's Laws). Let P and Q be statements.
(1) $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$.
(2) $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$.
Note: The truth tables you constructed on Group Assignment #1 prove the first part of this proposition.

1. Construct truth tables in order to prove that $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$.

Negating Statements: It is often important to be able to understand the meaning of the **negation** of a logical statement. Formally, we can just apply the negation connective to a statement. Although this gives us a logical representation that is technically correct, having a 'positively phrased" form of such a statement is generally more useful. Your textbook calls this a *useful denial* of the logical statement.

Example: Our goal is to find a useful negation of the statement $1 < x < 2$. To accomplish this, we first notice that this is a compound statement in disguise. This is really stating: $x > 1$ and $x < 2$. The formal negation of this is: $\neg[(x > 1) \wedge (x < 2)]$. Using DeMorgan's Law, we see this is equivalent to: $\neg(x > 1) \vee \neg(x < 2)$. Applying the "Trichotomy Property" of inequalities, we can rewrite this as: $(x \leq 1) \vee (x \geq 2)$. Thus, a useful denial of the original statement is the statement: $x \leq 1$ or $x \geq 2$.

2. Find a useful denial of the statement n is even or $n < 10$.

More logical connectives: Suppose P and Q represent statements. We define the **conditional** and **bi-conditional** logical connectives as follows:

- The *implication* or *conditional* statement $P \Rightarrow Q$ is the statement "If P then Q". We often call P the *hypothesis* or *antecedent* and Q the **conclusion** or *consequent*.
- The *bi-conditional* statement $P \Leftrightarrow Q$ is the statement "P if and only if Q". We can think of this statement as an abbreviated form of the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

3. **Note:** In order to create a truth table for the conditional, it is useful to think of $P \Rightarrow Q$ as a sort of "contract" or promise statement and to ask which case(s) would lead to a broken promise of contract. Consider the example statement: "If you mow the lawn then I will pay you $20". Complete the truth table for the conditional in the table provided below.

P	Q	$P \Rightarrow Q$
T	T	
T	F	
F	T	
F	F	

P	Q	$P \Leftrightarrow Q$
T	T	
T	F	
F	T	
F	F	

4. Use the fact that the bi-conditional is the conjunction of two conditional statements to write out its truth table in the table provided above.

Note: Several English phrases that are used to represent conditional and bi-conditional statements are given below. In each of these, P and Q play the same role as in the base symbolic statements $P \Rightarrow Q$ and $P \Leftrightarrow Q$

Alternate Forms of the Conditional

If P then Q
P only if Q
P is sufficient for Q
Q when P
Q if P
Q is necessary for P

Alternate Forms of the Bi-conditional

P if and only if Q
P iff Q
P is equivalent to Q
P exactly when Q
P is necessary and sufficient for Q

5. Determine whether or not the following statements are true or false.

(a) If $2 + 2 = 5$ then $2^3 = 8$

(b) If $2^3 = 8$ then $2 + 2 = 5$

(c) $2 + 2 = 5$ iff $\neg(2^3 = 8)$

Proposition 1.1.11 Let P and Q be statements.

(1) $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$ (2) $\neg(P \Rightarrow Q)$ is logically equivalent to $P \wedge (\neg Q)$ (3) $P \Leftrightarrow Q$ is true exactly when P and Q have the same truth value.

6. Use truth tables to prove either part (1) or (2) in this theorem (your choice).

7. Find a useful denial of the statement "If you mow the lawn then I will pay you $20".

8. Create a truth tables for the logical expressions: $\neg P \vee \neg Q$, and $\neg(P \vee Q)$

Note: There are several important statements related to the conditional $P \Rightarrow Q$. The textbook only gives two of these, but we will define all three.

Definition 1.1.13(plus addendum) Let P and Q be statements.

(1) The *converse* of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

(2) The *contrapositive* of $P \Rightarrow Q$ is the statement $(\neg Q) \Rightarrow (\neg P)$.

(3) The *inverse* of $P \Rightarrow Q$ is the statement $(\neg P) \Rightarrow (\neg Q)$.

9. Use truth tables to determine which of these three statements are logically equivalent to the original conditional statement $P \Rightarrow Q$.

10. Write the converse and contrapositive statements (in plain English) for the conditional statement "If you mow the lawn then I will pay you $20".

Quantifiers

Note: Recall that on Group Work #1, we determined that $x + 1 = 5$ is not a proposition because it is not possible to assign it a truth value as written. We consider this to be a **predicate** because it only becomes a proposition when the "free variable" x is assigned a specific value from the "universe" under consideration (think of this as the "domain" for the free variable).

If we think of the real numbers as the "universe" for free variable x, then we can define $P(x)$, the predicate statement $x + 1 = 5$. From this, the proposition $P(4)$ is True, since it represents $4 + 1 = 5$, while $P(5)$ is False, as it represents $5 + 1 = 5$.

Another way to change a predicate statement into a statement which has a truth value is to add a *quantifier.*

Definition 1.1.16: Let $\mathcal{U}$ be a universe under consideration and $P(x)$ be a predicate whose only free variable is x. Then the statements:

- "for all x, $P(x)$" [denoted $(\forall x)P(x)$]
- "there exists x such that $P(x)$" [denoted $(\exists x)P(x)$]

are statements that have truth values.

- The symbol $\forall$ is called the **universal quantifier**. The statement $(\forall x)P(x)$ is true exactly when each individual element a in the universe $\mathcal{U}$ has the property that $P(a)$ is true.
- The symbol $\exists$ is called the **existential quantifier**. The statement $(\exists x)P(x)$ is true exactly when the universe $\mathcal{U}$ contains at least one element a for which $P(a)$ is true.

11. Determine the truth value for each of the following statements:

    (a) For all real numbers r, $r \cdot 5 = 1$

    (b) There exists a real number r such that $r \cdot 5 = 1$

    (c) There exists a natural number n such that $n \cdot 5 = 1$

    (d) There exists a real number x such that for any real number y, $x \cdot y = y$.

Note: Familiarize yourself with following notation for some frequently used "universes" of numbers.

The **natural numbers** $\mathbb{N}$:	$\mathbb{N} = \{1, 2, 3, \cdots\}$.
The **integers** $\mathbb{Z}$:	$\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$.
The **rational numbers** $\mathbb{Q}$:	$\mathbb{Q} = \{x : \text{ there exist } a, b \in \mathbb{Z}, b \neq 0 \text{ such that } x = \frac{a}{b}\}$.
The **real numbers** $\mathbb{R}$:	(informally) $\mathbb{R} = \{x : x \text{ has a decimal expansion }\}$.

Negating Quantified Statements

Proposition 1.1.18 Let $P(x)$ be a predicate and let $\mathcal{U}$ be the intended universe. Then:

(1) $\neg(\forall x)P(x)$ is logically equivalent to $(\exists x)(\neg P(x))$; i.e., $\neg(\forall x \in \mathcal{U})P(x)$ is logically equivalent to $(\exists x \in U)(\neg P(x))$.

(2) $\neg(\exists x)P(x)$ is logically equivalent to $(\forall x)(\neg P(x))$; i.e., $\neg(\exists x \in \mathcal{U})P(x)$ is logically equivalent to $(\forall x \in U)(\neg P(x))$.

Note: The notation $(\forall x \in \mathcal{U})P(x)$ is an abbreviation for the statement $(\forall x)(x \in \mathcal{U} \Rightarrow P(x))$. Similarly, the notation $(\exists x \in \mathcal{U})P(x)$ is an abbreviation for the statement $(\exists x)(x \in \mathcal{U} \wedge P(x))$.

12. Find a "useful denial" for each of the following statements.

    (a) There exists a real number r such that $\sqrt{r} = -1$.

    (b) For all real numbers, if $x < 4$, then $x^2 < 16$.