Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 3 **Name:**______________________________
Due: Thursday, September 6th

Recall: Let $\mathcal{U}$ be a universe under consideration and $P(x)$ be a predicate whose only free variable is x.

- The symbol $\forall$ is called the **universal quantifier**. The statement $(\forall x)P(x)$ is true exactly when each individual element a in the universe $\mathcal{U}$ has the property that $P(a)$ is true.
- The symbol $\exists$ is called the **existential quantifier**. The statement $(\exists x)P(x)$ is true exactly when the universe $\mathcal{U}$ contains at least one element a for which $P(a)$ is true.

The symbolization $\exists!$ represents another important quantifier, the **uniqueness quantifier**. The statement $(\exists! x)P(x)$ is true exactly when the universe $\mathcal{U}$ contains **exactly one** element a for which $P(a)$ is true.

1. Describe two different ways that a statement of the form $(\exists! x)P(x)$ can fail to be true.

2. Let $\mathcal{U} = \mathbb{R}$. Determine the truth value for each of the following statements:

   (a) $(\exists! x)(x + 2 > 0)$ (b) $(\forall x)(\exists! y)(x + y = 0)$ (c) $(\exists! x)(\forall y)(xy = 0)$ (d) $(\forall x)(\exists! y)(xy = 1)$

Hidden Quantifiers: Let $\mathcal{U} = \mathbb{Z} = \{\cdots - 3, -2, -1, 0, 1, 2, 3, \cdots\}$. Consider the statement "If n is odd, then n^2 is also odd". Notice that this is actually a quantified statement, even though it is not explicitly stated that way, since it is making a general claim about the way integers behave.

3. Carefully rewrite the statement "If n is odd, then n^2 is also odd" as a carefully phrased quantified statement.

4. Find a useful denial (a simplified negation) of the original statement.

Definition 1.1.20 A **tautology** is a proposition that is true for every possible assignment of truth values to the statement letters that occur within it. A **contradiction** is a proposition that is false for every possible assignment of truth values. A **contingency** is a proposition that is true for at least one truth value assignment and false for at least one truth value assignment.

5. For each proposition given below, state the total number of possible truth value assignments. Then determine whether it is a tautology, a contradiction, or a contingency. [Do you need to build a complete truth table?]

   (a) $P \vee \neg P$ (b) $P \wedge \neg P$ (c) $P \wedge (Q \vee R)$ (d) $P \vee Q \vee \neg R \wedge S$.

Mathematical Proof

Now that we have established a good foundation in mathematical logic, we will now turn our attention to learning to build sound mathematical arguments (these are commonly called "proofs"). Informally, a *mathematical proof* is a logical argument that establishes the truth of a mathematical statement.

Example: Suppose that we have the following two established facts (such facts are often called **premises**).

- If I stay up late, then I will be tired in the morning.
- I stayed up late.

6. Give a valid conclusion that follows from these two premises.

If you found the correct valid conclusion for the example given above, chances are, you (consciously or unconsciously) used a famous and important *rule of deduction* called *modus ponens.* The form of this rule of deduction is as follows:

modus ponens

$$\begin{array}{l} P \rightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

7. Use formal logic to verify that modus ponens is a valid logical argument by proving that the associated logical statement $[(P \Rightarrow Q) \wedge P] \Rightarrow Q$ is a tautology.

Now that we know that modus ponens is valid, this gives us an argument strategy that we can use whenever we want either by itself or as part of a longer mathematical argument.

Note: In general, the method that we use to prove a mathematical statement will depend on the logical form of the underlying statement. Because of this, in any attempt to create a proof, our first job will be to **correctly identify the underlying form** of the statement that we are trying to prove.

Direct Proofs

Here are three foundational proof strategies that we will use on a consistent basis.

To prove that a statement of the form $P \Rightarrow Q$ is true (directly)
• Begin by stating: "Suppose that P is true."
• Use definitions and established mathematical principles to demonstrate that Q is true.

To prove that a statement of the form $(\forall x)P(x)$ is true (directly)
• Begin by stating "Let x be an arbitrary (but now *fixed*) element of the universe."
• Use definitions and established mathematical principles to demonstrate that $P(x)$ is true.

To prove that a statement of the form $(\exists x)P(x)$ is true (directly)
• Find a specific element a in the universe such that $P(a)$ is true.
• That is, identify a specific element of the universe a and demonstrate that $P(a)$ is true for that element.

Note: As mentioned above, mathematical definitions are another type of key building blocks that we will use to help construct mathematical arguments. Note that even though we do not always phrase them explicitly this way, mathematical definitions are always considered to be **if and only if** statements. Here are a few to get us started.

Definition 1.2.1 Let $n \in \mathbb{Z}$.

(1) n is **even** if there exists an integer k such that $n = 2k$; i.e., n is even if and only if $(\exists k \in \mathbb{Z})[n = 2k]$.

(2) n is **odd** if there exists an integer k such that $n = 2k + 1$; i.e., n is odd if and only if $(\exists k \in \mathbb{Z})[n = 2k + 1]$.

(3) n is **prime** if $n > 1$ and the only positive integer factors of n are 1 and n; i.e., n is prime if $n > 1$ and $(\forall a, b \in \mathbb{N})[n = ab \Rightarrow (a = 1 \text{ or } b = 1)]$.

If possible, prove each of the following statements.

8. The sum of two odd integers is even.

9. If n is even, then n^2 is even.

10. If n is odd, then n^2 is odd.

11. There are two consecutive positive integers that are both prime.

12. If n is odd, and m is even then mn is even.

Presentation Problem: Hazel, Robyn, and Tim all enjoy playing chess. Although all three are good players, Hazel is the best player, followed by Robyn, and Tim is the weakest player of the three. One evening, Hazel and Robyn want to play chess, but Tim is getting tired of losing. To encourage him to play, Hazel makes the following suggestion to Tim: "Lets play three games of chess. You must play against either Robyn or I and you must alternate between the two of us each game. If you succeed in winning two games in a row, then I will give you $20."

Tim asks: "Who do I play first, you or Robyn?"

Hazel replies: "That is entirely up to you."

To maximize his chance of winning two games in a row, which person should Tim choose to play against first?