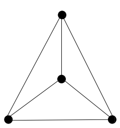
Math 311
Project 5 Handout

Instructions: This handout is designed to describe a project on one of the "Additional Topics" that may to choose to complete in order to satisfy the Project component of your grade in the course. As noted in the Course Policies Handout, you can earn 25 points (or more) based on the progress you make on these projects. To help make sure that you have time to work on these, you will be given time in class to work on these rather than our standard Daily Group Work assignments during the next three class periods (starting on Thursday, November 29th). You are encouraged to work in groups on these projects. You should turn in **three** projects for grading. One should be completed by Tuesday, December 4th. Your second is due on Thursday, December 6th, and your third project must be completed by Tuesday, December 11th. You will have the opportunity to earn up to 10-12 points on each of these projects. Any points beyond 25 points will count as extra credit.

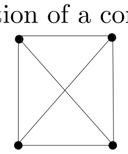
Project 5: Planar Graphs

In this project, you will investigate when it is possible to draw a graph in the plane without any edges crossing and when it is not possible to do so. This includes the famous "three utility lines" problem. The fundamental question that we will ask is: "When is it possible to draw a graph so that none of its edges cross?". Since there are many possible ways to draw the same graph, a better way to phrase this question might be: "When is it possible to find at least one way to represent a graph in a plane so that none of its edges cross?"

To see why we need to think about both the graph itself (the number of edges and vertices and which pairs of vertices are connected by an edge) and how we choose to draw the graph (i.e. how we position the vertices relative on one another) can be seen in the following example. Here are two different ways to draw K_4, the complete graph on 4 vertices (see Project 4 for the definition of a complete graph).



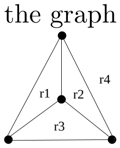
K_4



also K_4

Notice that the first way of representing this graph in the plane has on edges crossing (the edges only meet at vertices) while the second drawing does have a pair of edges that cross. When a drawing like the first one exists for a graph, we say that the graph is **planar** and we call the drawing a **planar representation** of the graph.

When a graph has a planar representation, the representation divides the plane into regions. For example, in our planar representation of K_4, the graph has 4 vertices, 6 edges, and 4 regions (see the figure below). Note that the area "outside" the graph counts as a region.



K_4

Euler's formula describes the relationship between the number of vertices, edges, and regions in any simple planar graph.
Euler's Formula: Let Γ be a planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of Γ. Then $r = e - v + 2$.

1. (2 points) Draw two different examples of planar simple graphs and verify that Euler's Formula holds for these graphs. Your examples must involve at least 6 vertices and must divide the plane up into at least 4 regions.

2. (2 points) Determine (with proof) whether or not the graph $K_{3,3}$ (the complete bipartite graph with two vertex classes of size three) is planar. In doing so, you will determine whether or not it is possible connect three houses on the same side of the street to three separate utilities (cable, water, and electricity) without having two of the utility lines cross.

3. (2 points) Determine whether or not Euler's Formula holds for graphs that have loop edges.

4. (2 points) Determine whether or not Euler's Formula holds for graphs that have multi edges.

5. (2 points) Prove that if Γ is a simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$. [Hint: use the fact that in a simple graph, at least three edges are required in order to form a region and the fact that each edge in the graph occurs within the boundary of a region exactly twice].

6. (2 points) A coloring of a graph is a an assignment of colors to the vertices of the graph in such a way that any two vertices that are connected by an edge are assigned different colors. Find examples of planar graphs that require 2, 3, and 4 different colors (the idea here is to use the minimum number of colors necessary). Either find an example of a planar graph that requires 5 different colors or explain why no such graph exists.