

**Example:** Note that  $\int_0^{1.5} \tan x \, dx = -\ln |\cos x| \Big|_0^{1.5} \approx 2.648783654$ .

We will demonstrate the method of adaptive quadrature. Given  $f(x) = \tan x$ , we begin by taking  $h = 0.75$ .

Our goal is to approximate  $\int_0^{1.5} \tan x \, dx$  to within an error tolerance of  $\varepsilon = 0.01$ .

**Step 1:**  $S(0, 1.5) = \frac{0.75}{3} [f(0) + 4f(0.75) + f(1.5)] \approx 4.456951447$

$S(0, 0.75) = \frac{0.75}{6} [f(0) + 4f(0.375) + f(0.75)] \approx 0.313262845$

$S(0.75, 1.5) = \frac{0.75}{6} [f(0.75) + 4f(1.125) + f(1.5)] \approx 2.925412689$ .

Thus  $|S(0, 1.5) - S(0, 0.75) - S(0.75, 1.5)| \approx 1.218276$ , which is **not** less than  $15\varepsilon = 0.15$ . Hence we need to subdivide both half intervals and compute again.

**Step 2a:** (Left Half)  $S(0, 0.375) = \frac{0.75}{12} [f(0) + 4f(0.1875) + f(0.375)] \approx 0.0720338137$

$S(0.375, 0.75) = \frac{0.75}{12} [f(0.375) + 4f(0.5625) + f(0.75)] \approx .2404358582$ .

Thus  $|S(0, 0.75) - S(0, 0.375) - S(0.375, 0.75)| \approx 0.000793173$ , which is less than  $15 \cdot \frac{\varepsilon}{2} = 0.075$ . Hence we **do not** need to subdivide this half interval a second time.

**Step 2b:** (Right Half)  $S(0.75, 1.125) = \frac{0.75}{12} [f(0.75) + 4f(0.9375) + f(1.125)] \approx 0.5295284776$

$S(1.125, 1.50) = \frac{0.75}{12} [f(1.125) + 4f(1.3125) + f(1.50)] \approx 1.958383993$ .

Thus  $|S(0.75, 1.50) - S(0.75, 1.125) - S(1.125, 1.50)| \approx 0.4375\dots$ , which is **not** less than  $15 \cdot \frac{\varepsilon}{2} = 0.075$ . Hence we need to subdivide this half interval again.

**Step 3a:** (Left Half)  $S(0.75, 0.9375) = \frac{0.75}{24} [f(0.75) + 4f(0.84375) + f(0.9375)] \approx 0.21218778$

$S(0.9375, 1.125) = \frac{0.75}{24} [f(0.9375) + 4f(1.03125) + f(1.125)] \approx 0.316703867$ .

Thus  $|S(0.75, 1.125) - S(0.75, 0.9375) - S(0.9375, 1.125)| \approx 0.000636$ , which is less than  $15 \cdot \frac{\varepsilon}{4} = 0.0375$ . Hence we **do not** need to subdivide this half interval again.

**Step 3b:** (Right Half)  $S(1.125, 1.3125) = \frac{0.75}{24} [f(1.125) + 4f(1.21875) + f(1.3125)] \approx 0.523950957$

$S(1.3125, 1.50) = \frac{0.75}{24} [f(1.3125) + 4f(1.40625) + f(1.50)] \approx 1.311747793$ .

Thus  $|S(1.125, 1.50) - S(1.125, 1.3125) - S(1.3125, 1.50)| \approx 0.122\dots$ , which is **not** less than  $15 \cdot \frac{\varepsilon}{4} = 0.0375$ . Hence we do need to subdivide this half interval again.

**Step 4a:** (Left Half)  $S(1.125, 1.21875) = \frac{0.75}{48} [f(1.125) + 4f(1.171875) + f(1.21875)] \approx 0.223502989$

$S(1.21875, 1.3125) = \frac{0.75}{48} [f(1.21875) + 4f(1.265625) + f(1.3125)] \approx 0.30081106$ .

Thus  $|S(1.125, 1.3125) - S(1.125, 1.21875) - S(1.21875, 1.3125)| \approx 0.00036\dots$ , which is less than  $15 \cdot \frac{\varepsilon}{8} = 0.01875$ . Hence we **do not** need to subdivide this half interval again.

**Step 4b:** (Right Half)  $S(1.3125, 1.40625) = \frac{0.75}{48} [f(1.3125) + 4f(1.359375) + f(1.40625)] \approx 0.4441432269$

$S(1.40625, 1.50) = \frac{0.75}{48} [f(1.40625) + 4f(1.453125) + f(1.50)] \approx 0.8431208694$ .

Thus  $|S(1.3125, 1.50) - S(1.3125, 1.40625) - S(1.40625, 1.50)| \approx 0.0244\dots$ , which is **not** less than  $15 \cdot \frac{\varepsilon}{8} = 0.01875$ . Hence we need to subdivide this half interval again.

**Step 5a:** (Left Half)  $S(1.3125, 1.359375) = \frac{0.75}{96} [f(1.3125) + 4f(1.3359375) + f(1.359375)] \approx 0.196573833$

$S(1.359375, 1.40625) = \frac{0.75}{96} [f(1.359375) + 4f(1.3828125) + f(1.40625)] \approx 0.247724715$ .

Thus  $|S(1.3125, 1.40625) - S(1.3125, 1.359375) - S(1.359375, 1.40625)| \approx 0.00014\dots$ , which is less than  $15 \cdot \frac{\varepsilon}{16} = 0.009375$ . Hence we **do not** need to subdivide this half interval again.

**Step 5b:** (Right Half)  $S(1.40625, 1.453125) = \frac{0.75}{96} [f(1.40625) + 4f(1.4296875) + f(1.453125)] \approx 0.333124023$

$S(1.453125, 1.50) = \frac{0.75}{96} [f(1.453125) + 4f(1.4765625) + f(1.50)] \approx 0.506892887$ .

Thus  $|S(1.40625, 1.50) - S(1.40625, 1.453125) - S(1.453125, 1.50)| \approx 0.0031\dots$ , which is less than  $15 \cdot \frac{\varepsilon}{16} = 0.009375$ . Hence we **do not** need to subdivide this half interval again, so we are done computing half intervals.

**Step 6:** To finish, we add up the approximation associated with each previously computed interval:

$$\begin{aligned} \int_0^{1.5} \tan x \, dx &\approx S(0, 0.375) + S(0.375, 0.75) + S(0.75, 0.9375) + S(0.9375, 1.125) + S(1.125, 1.21875) + S(1.21875, 1.3125) + \\ &S(1.3125, 1.359375) + S(1.359375, 1.40625) + S(1.40625, 1.453125) + S(1.453125, 1.5) \\ &\approx 0.072033814 + 0.2404358582 + 0.21218778 + 0.316703867 + 0.223502989 + 0.30081106 + 0.196573833 + 0.247724715 + \\ &0.333124023 + 0.506892887 \\ &= 2.649260871 \text{ (Notice that this does approximate the original definite itegral to within our desired error tolerance).} \end{aligned}$$