

**Recall:** The recursion for Higher Order Taylor Methods to approximate solutions to a well posed initial value problem of the form  $\frac{dy}{dt} = f(t, y)$ ,  $y(a) = \alpha$  over  $D = \{(t, y) : a \leq t \leq b, y \in \mathbb{R}\}$  is given by:

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h \cdot T^{(n)}(t_i, w_i) \text{ where } T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \dots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i, w_i).$$

**Example:** Consider the well posed IVP given by  $\frac{dy}{dt} = t^3 + t - y$ ,  $0 \leq t \leq 2$ ,  $y(0) = 1$ . Use Taylor's Methods of order 2 and 4 to approximate  $y(2)$  using step size  $h = 0.5$ .

We begin by computing the derivatives necessary to find the recursions for second and fourth order Taylor's Methods.

$$y'(t) = f(t, y) = t^3 + t - y$$

$$y''(t) = f'(t, y) = 3t^2 + 1 - y' = 3t^2 + 1 - (t^3 + t - y) = -t^3 + 3t^2 - t + 1 + y$$

$$y'''(t) = f''(t, y) = -3t^2 + 6t - 1 + y' = -3t^2 + 6t - 1 + t^3 + t - y = t^3 - 3t^2 + 7t - 1 - y$$

$$y^{(4)} = f'''(t, y) = 3t^2 - 6t + 7 - y' = 3t^2 - 6t + 7 - (t^3 + t - y) = -t^3 + 3t^2 - 7t + 7 + y$$

From these, we have the following:

$$T^{(2)}(t_i, w_i) = (t_i^3 + t_i - w_i) + \frac{h}{2}(-t_i^3 + 3t_i^2 - t_i + 1 + w_i) = t_i^3 + t_i - w_i - \frac{1}{2}ht_i^3 + \frac{3}{2}ht_i^2 - \frac{1}{2}ht_i + \frac{1}{2}h + \frac{1}{2}hw_i$$

$$T^{(4)}(t_i, w_i) = (t_i^3 + t_i - w_i) + \frac{h}{2}(-t_i^3 + 3t_i^2 - t_i + 1 + w_i) + \frac{h^2}{6}(t_i^3 - 3t_i^2 + 7t_i - 1 - w_i) + \frac{h^3}{24}(-t_i^3 + 3t_i^2 - 7t_i + 7 + w_i) = t_i^3 + t_i - w_i - \frac{1}{2}ht_i^3 + \frac{3}{2}ht_i^2 - \frac{1}{2}ht_i + \frac{1}{2}h + \frac{1}{2}hw_i + \frac{1}{6}h^2t_i^3 - \frac{1}{2}h^2t_i^2 + \frac{7}{6}h^2t_i - \frac{1}{6}h^2 - \frac{1}{6}h^2w_i - \frac{1}{24}h^3t_i^3 + \frac{1}{8}h^3t_i^2 - \frac{7}{24}h^3t_i + \frac{7}{24}h^3 + \frac{1}{24}h^3w_i$$

Using these, we have the following:

**Second Order Taylor:**

$$w_0 = 1$$

$$w_{i+1} = w_i + h \cdot T^{(2)}(t_i, w_i) = w_i + (0.5) \cdot (t_i^3 + t_i - w_i - \frac{1}{2}(0.5)t_i^3 + \frac{3}{2}(0.5)t_i^2 - \frac{1}{2}(0.5)t_i + \frac{1}{2}(0.5) + \frac{1}{2}(0.5)w_i)$$

$$w_1 = 0.75$$

$$w_2 = 0.9218750000$$

$$w_3 = 1.826171875$$

$$w_4 = 3.938232422$$

**Fourth Order Taylor:**

$$w_0 = 1$$

$$w_{i+1} = w_i + h \cdot T^{(4)}(t_i, w_i) = w_i + (0.5) \cdot \left( t_i^3 + t_i - w_i - \frac{1}{2}(0.5)t_i^3 + \frac{3}{2}(0.5)t_i^2 - \frac{1}{2}(0.5)t_i + \frac{1}{2}(0.5) + \frac{1}{2}(0.5)w_i + \frac{1}{6}(0.5)^2t_i^3 - \frac{1}{2}(0.5)^2t_i^2 + \frac{7}{6}(0.5)^2t_i - \frac{1}{6}(0.5)^2 - \frac{1}{6}(0.5)^2w_i - \frac{1}{24}(0.5)^3t_i^3 + \frac{1}{8}(0.5)^3t_i^2 - \frac{7}{24}(0.5)^3t_i + \frac{7}{24}(0.5)^3 + \frac{1}{24}(0.5)^3w_i \right)$$

$$w_1 = 0.7291666667$$

$$w_2 = 0.9453667535$$

$$w_3 = 1.912162639$$

$$w_4 = 4.084398164$$