Math 450

Section 2.6: *Müller's Method Example*

Recall: When using *Müller's Method*, we are given three initial approximations based at the x values p_0, p_1, and p_2. So we are given a function $f(x)$ that we want to find a root of along with initial data $(p_0, f(p_0))$, $(p_1, f(p_1))$, and $(p_2, f(p_2))$. We generally label these so that $p_0 < p_1$, with $p_2 \in (p_0, p_1)$ and with $f(p_0) \cdot f(p_1) < 0$. This method generates a new approximation p_3 based on the given initial data by finding a root of the quadratic polynomial $P_2(x) = ax^2 + bx + c$ passing through all three initial data points.

To do this, we set $P_2(x) = \dfrac{(x-p_1)(x-p_2)}{(p_0-p_1)(p_0-p_2)} \cdot f(p_0) + \dfrac{(x-p_0)(x-p_2)}{(p_1-p_0)(p_1-p_2)} \cdot f(p_1) + \dfrac{(x-p_0)(x-p_1)}{(p_2-p_0)(p_2-p_1)} \cdot f(p_2)$.

Note that is it fairly straightforward to verify that this function passes through the given initial points (just evaluate at $x = p_0$, at $x = p_1$ and at $x = p_2$ and verify that this results in the correct y-value).

By some tedious algebra, one can show that:

- $c = f(p_2)$
- $b = \dfrac{(p_0-p_2)^2[f(p_1)-f(p_2)] - (p_1-p_2)^2[f(p_0)-f(p_2)]}{(p_0-p_2)(p_1-p_2)(p_0-p_1)}$
- $a = \dfrac{(p_1-p_2)[f(p_0)-f(p_2)] - (p_0-p_2)[f(p_1)-f(p_2)]}{(p_0-p_2)(p_1-p_2)(p_0-p_1)}$

Once we have this, we can use the quadratic formula (pick your favorite of the two forms that we discussed last chapter) and solve for the root p_3 between p_0 and p_1 (by picking the right sign in the QF). We then use p_2. p_3, and whichever of p_0 and p_1 evaluate to have the opposite sign of $f(p_2)$ and restart with these three initial data values.

Example: Let $f(x) = 3x^3 - 13x^2 + 19x - 5$. Let $p_0 = -1$, $p_1 = 1$, and $p_2 = 0$. Note that $p_2 \in [-1, 1]$ and $f(p_0) \cdot f(p_2) = f(-1) \cdot f(1) = (-40)(4) = -160 < 0$.

Computing carefully, $P_2(x) = \dfrac{x(x-1)}{2}(-40) + \dfrac{(x)(x+1)}{2}(4) + \dfrac{(x+1)(x-1)}{(1)(-1)}(-5) = -20(x^2-x) + 2(x^2+x) + 5(x^2-1) = -13x^2 + 22x - 5$.

Using the alternative form of the quadratic formula, let $p_3 = \dfrac{10}{22 + \sqrt{22^2 = 4(-13)(-5)}} = \dfrac{10}{22+\sqrt{224}} \approx 0.270514248$.

Since $f(p_1)$ and $f(p_2)$ have the opposite signs, We set our new $p_1 = 1$ (our old p_1), $p_2 = 0$ (our old p_2) and we set $p_3 = 0.2705$ (not sure why we are only keeping 4 digits, but we have to round somewhere) and iterate the same procedure.

Here, $P_2(x) = \dfrac{(x-1)(x-0.2705)}{0.2705}(-5) + \dfrac{(x)(x-0.2705)}{0.7295}(4) + \dfrac{x(x-1)}{0.2705}(-0.7523) \approx -9.18868x^2 + 18.18868x - 5$.

Hence $p_4 = \dfrac{10}{18.18868 + \sqrt{(18.18868^2 - 4(-9.18868)(-5)}} \approx 0.32986631$

Continuing in this manner, we obtain more and more accurate approximations of the actual root (the root of $f(x)$ within our initial interval).