Review of Polynomial Functions

The Fundamental Theorem of Algebra: (Gauss - 1799)
A polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $n \geq 1$ and $a_i \in \mathbb{C}$ has at least one zero in $\mathbb{C}$ where $\mathbb{C}$ is the set of complex numbers.

We will assume that all polynomials have real coefficients. That is, each $a_i \in \mathbb{R} \subset \mathbb{C}$.

The Remainder Theorem: If a polynomial $P^n(x)$ is divided by $x - a$ until a constant remainder, R, is obtained, then $R = P_n(a)$.

Proof. $P_n(x) = (x-a)Q_{n-1}(x) + R$ for all x. Therefore, $P_n(a) = (a-a)Q_{n-1}(a) + R = R$. □

The Factor Theorem: Assume $P_n(x)$ is a polynomial. $P_n(p) = 0$ iff $x - p$ is a factor.

Proof. ($\Rightarrow$) Dividing $P_n(x)$ by $x - p$ yields $P_n(x) = (x-p)Q_{n-1}(x) + R$. Since $P_n(p) = 0$ by the Remainder Theorem, $R = 0$. Therefore, $P_n(x) = (x-p)Q_{n-1}(x)$. Hence $x - p$ is a factor of $P_n(x)$.
($\Leftarrow$) If $x - p$ is a factor of $P_n(x)$, then there is a function $Q_{n-1}(x)$ such that $P_n(x) = (x-p)Q_{n-1}(x)$. Thus $P_n(p) = (p-p)Q_{n-1}(p) = 0$. □

Corollary: If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $n \geq 1$, then there exists unique constants (possibly complex) and positive integers $m_1,\ m_2,\ \ldots,\ m_k$ such that

$$\sum_{i=1}^{k} m_i = n \quad \text{and} \quad P(x) = a_n(x-x_1)^{m_1}(x-x_2)^{m_2}\cdots(x-x_k)^{m_k}.$$

Proof. From the Fundamental Theorem of Algebra and the Remainder Theorem applied to $P_n(x)$ there is an $x_1 \in \mathbb{C}$ and a polynomial $Q_{n-1}(x)$ such that $P_n(x_1) = 0$ and $P_n(x) = (x-x_1)Q_{n-1}(x)$. If $n - 1 \geq 1$, we repeat the process on $Q_{n-1}(x)$ and express $Q_{n-1}(x)$ as $Q_{n-1}(x) = (x-x_2)Q_{n-2}(x)$. Continuing inductively in this manner, there exist $x_3, \ldots, x_n$ such that

$$P_n(x) = (x-x_1)(x-x_2)\cdots(x-x_n)Q_0(x).$$

Note $Q_0(x)$ is a constant function. In order for the product, on the right, to equal $P_n(x)$, it is necessary that $Q_0(x) = a_n$. Now gather the like factors and relabel, if necessary, we obtain the desired result. □

Theorem: Assume P and Q are polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$ with $k > n$ are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, \ldots, k$, then $P(x) = Q(x)$ for all x.

Proof. Let $D(x) = P(x) - Q(x)$. Then the degree of $D(x)$ is less than or equal to n. Also, $D(x_i) = P(x_i) - Q(x_i) = 0$ for $i = 1, 2, \ldots, k$ where $k > n$. The only polynomial of degree less than or equal to n with $k > n$ zeros is the zero polynomial $D(x) \equiv 0$. Hence $P(x) \equiv Q(x)$. □

Theorem: If $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and $Q(x) = b_n x^n + \cdots + a_1 x + a_0$ are polynomials of degree at most n and if $P_n(x_i) = Q_n(x_i)$ for distinct numbers $x_1, \ldots, x_k$ where $k > n$, then $a_i = b_i$ for all $i = 0, 1,\ 2,\ \ldots, n$.

Proof. Let $D(x) = P_n(x) - Q_n(x) = (a_n - b_n)x^n + \cdots + (a_0 - b_0)$. Note $D(x_i) = P(x_i) - Q(x_i) = 0$ for all $i = 1, \ldots, k$ and $D(x)$ is a polynomial of degree at most n with $k > n$ zeros. By the previous theorem, this is not possible unless the degree of D is less than 1 or is nonexistent. Thus $D(x) \equiv 0$. Therefore, each $a_1 - b_i = 0$. Hence $a_i = b_i$ for all $i = 0, 1, \ldots, n$. □

Theorem: Assume $P(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial with real coefficients. If $a + bi$ is a root of P, then $a - bi$ is also a root of P.

Corollary: Any odd degree polynomial with real coefficients has at least one real root.

The Rational Root Theorem: If $\frac{p}{q}$ is a rational root (simplified form) of the polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ where each a_i is an integer, then p is a divisor of a_0 and q is a divisor of a_n.

Proof. Assume $\frac{p}{q}$ is a simplified rational root of the polynomial $P(x)$. Thus

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$
$$a_n\left(\frac{p^n}{q^n}\right) + a_{n-1}\left(\frac{p^{n-1}}{q^{n-1}}\right) + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$
$$a_n p^n + a_{n-1} q p^{n-1} + \cdots + a_1 q^{n-1} p + a_0 q^n = 0$$
$$a_n p^n + a_{n-1} q p^{n-1} + \cdots + a_1 q^{n-1} p = -a_0 q^n.$$

Note that p divides the left-hand side and q divides the right-hand side. Since p has no factors in common with q, p divides a_0. Rewriting as

$$a_{n-1} q p^{n-1} + \cdots + a_1 q^{n-1} p + a_0 q^n = -a_n p^n.$$

Note that q divides the left-hand side and p divides the right-hand side. Since q has no factors in common with p, q divides a_n. ☐