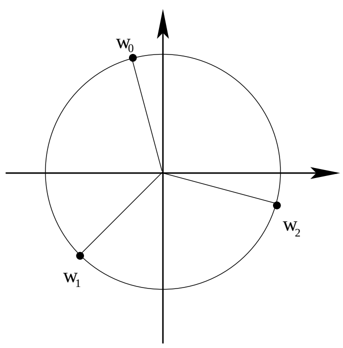
Math 143

Finding Roots of Complex Numbers

Suppose that $z = a + bi$ is a complex number, and we wish to find the n^{th} roots of z.
Further suppose that the polar form of z is given by: $z = rcis\theta$.
By Applying DeMoivre's Theorem, we know that if we put a complex number w into polar form: $w = s \cdot cis(\alpha)$, then $w^n = s^n cis(n\alpha)$. From this, if w is an n^{th} root of z, we must have that $w^n = z$, therefore, $s^n cis(n\alpha) = rcis\theta$.
From this, we know that $s = \sqrt[n]{r}$, and $\alpha = \frac{\theta}{n}$ gives an n^{th} root of z. This is a pretty straightforeward computation for any specific example.
What is a bit less obvious is that $w = \sqrt[n]{r}$ is not the only n^{th} root of z. In fact, z has exactly n *distinct* n^{th} roots. To clarify this, when we say *distinct* n^{th} roots, we mean that their standard forms are distinct. Of course, due to the fact that a single complex number has several polar forms (infinitely many, in fact), we need to use the standard forms when deciding whether n^{th} roots given in polar form are actually the same.
To find all n distinct n^{th} roots of z, we need to consider **all** angles that satisfy the equation $n\alpha = \theta + 2\pi k$, since whenver this is true, we get $w^n = z$. Therefore, we are thinking about all the angles α for which $\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}$ for some integer k. As it turns out, once we look at the αs we find after going through $k = 0, 1, 2, ..., n-1$, we start getting "repeated values" – that is, polar forms whose standard forms are the same as roots that we already found.
That is, we get the n distinct n^{th} roots of a complex number z by taking $s = \sqrt[n]{r}$, and $\alpha_k = \frac{\theta}{n} + \frac{2\pi k}{n}$ for $k = 0, 1, 2, ..., n-1$.
Example: Find the cube roots of $z = 1 - i$.
Solution:
First notice that $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, and θ is the angle with reference angle $\frac{\pi}{4}$ in the 4th quadrant. Thus the polar form of z is given by :$z = \sqrt{2}cis(\frac{7\pi}{4})$
Therefore, to find the cube roots of z, we first notice that $s = \sqrt[3]{\sqrt{2}} = \sqrt[6]{2}$ gives the modulus of any cube root of z.
Next, we need to find the various angles that work:
$\alpha_0 = \frac{\frac{7\pi}{4}}{3} + 0 = \frac{7\pi}{12}$ gives one cube root. This is a bit nicer to represent in degrees: $\alpha_0 = \frac{7 \cdot 180}{12} = (7)(15) = 105°$
similarly, $\alpha_1 = 105° + \frac{360}{3} = 105 + 120 = 225°$
and $\alpha_2 = 105° + 2 \cdot \frac{360}{3} = 105 + 240 = 345°$
Thus, we now see that the cube roots of $z = 1 - i$ are:
$w_0 = \sqrt[6]{2}cis(105°)$
$w_1 = \sqrt[6]{2}cis(225°)$
$w_2 = \sqrt[6]{2}cis(345°)$
Here is a graphical representation of the cube roots of $z = 1 - i$



Note: To get a better idea of exactly what is going on here, try computing w_0^3, w_1^3, and w_2^3. You should get z for each of these, but only after simplfying the polar form you obtain when multiplying these out by subtracting the appropriate multiple of 360°. You should also notice that $\alpha_3 = 105° + 3 \cdot 120° = 105° + 360° \equiv 105°$, so we do not get any aditional cube roots by continuing the process used above.