Math 143
Exam 3 - Solutions **Name:**______________________________
03/23/2007

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. I will give credit to each problem proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Be sure to simplify answers when possible. Also, make sure to follow directions carefully on each problem.

1. (7 points each) Find the *exact* value of the following:

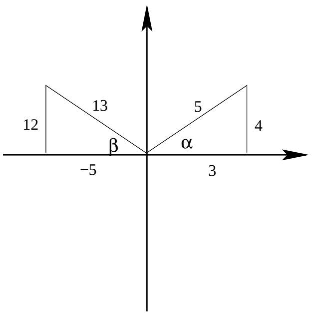
   (a) $\sin(105°)$

   $\sin(105°) = \sin(60° + 45°) = \sin(60°)\cos(45°) + \sin(45°)\cos(60°) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$
   $= \left(\frac{\sqrt{6}}{4}\right) + \left(\frac{\sqrt{2}}{4}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

   (b) $\cos(22.5°)$

   $$\cos(22.5°) = \cos\left(\frac{45°}{2}\right) = +\sqrt{\frac{1+\cos 45°}{2}} = +\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = +\sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}}$$
   $$= \frac{\sqrt{2+\sqrt{2}}}{2}$$

2. (7 points each) Given that $\sec\alpha = \frac{5}{3}$ and $\cos\beta = -\frac{5}{13}$, where α is in the first quadrant, and β is in the second quadrant, find *exact* values of:



   (a) $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) = \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right) = \left(-\frac{20}{65}\right) + \left(\frac{36}{65}\right) = \frac{16}{65}$

   (b) $\cos\left(\frac{\beta}{2}\right) = +\sqrt{\frac{1+\cos\beta}{2}} = +\sqrt{\frac{1-\frac{5}{13}}{2}} = +\sqrt{\frac{\frac{13}{13}-\frac{5}{13}}{2}} = \sqrt{\frac{\frac{8}{13}}{2}} = \sqrt{\frac{8}{26}} = \sqrt{\frac{4}{13}} = \frac{2\sqrt{13}}{13}$

   Note: we took the positive value since if β is in the second quadrant, then $90 < \beta < 180$, hence $45 < \frac{\beta}{2} < 90$, so $\frac{\beta}{2}$ is in the first quadrant.

3. (8 points each) Verify the following identities by transforming the left hand side into the right hand side:

(a) $\sin\left(\theta - \dfrac{3\pi}{2}\right) = \cos\theta$

$$\sin\left(\theta - \frac{3\pi}{2}\right) = \sin(\theta)\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right)\cos(\theta) = (\sin\theta)(0) - (-1)(\cos\theta) = \cos\theta$$

(b) $\dfrac{\cos x}{1+\sin x} = \sec x - \tan x$

$$\frac{\cos x}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x} = \frac{\cos x - \sin x\cos x}{1-\sin^2 x} = \frac{\cos x(1-\sin x)}{\cos^2 x} = \frac{1-\sin x}{\cos x} = \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$= \sec x - \tan x$$

(c) $\sec t - \cos t = \sin t \tan t$

$$\sec t - \cos t = \frac{1}{\cos t} - \cos t = \frac{1}{\cos t} - \frac{\cos^2 t}{\cos t} = \frac{1-\cos^2 t}{\cos t} = \frac{\sin^2 t}{\cos t} = \sin t\frac{\sin t}{\cos t} = \sin t\tan t$$

(d) $\sin 4t = 4\sin t\cos t(1-2\sin^2 t)$

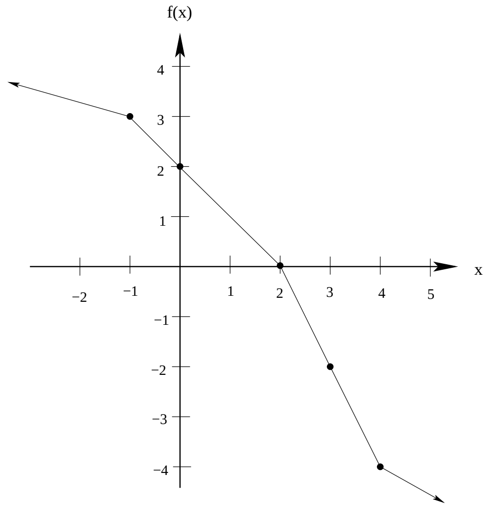
$$\sin 4t = \sin(2\cdot 2t) = 2\sin 2t\cos 2t = 2(2\sin t\cos t)(1-2\sin^2 t) = 4\sin t\cos t(1-2\sin^2 t)$$

4. (9 points each) Find *exact* solutions to the following equations with $0 \le \theta < 2\pi$.

(a) $\sin(2x + \frac{\pi}{6}) = -\frac{1}{2}$
Let $\theta = 2x + \frac{\pi}{6}$. Then $\sin(\theta) = -\frac{1}{2}$, so $\theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$.
That is, either $2x + \frac{\pi}{6} = \frac{7\pi}{6} + 2\pi n$ or $2x + \frac{\pi}{6} = \frac{11\pi}{6} + 2\pi n$.
If $2x + \frac{\pi}{6} = \frac{7\pi}{6} + 2\pi n$, then $2x = \pi + 2\pi n$, so $x = \frac{\pi}{2} + \pi n$.
On the other hand, if $2x + \frac{\pi}{6} = \frac{11\pi}{6} + 2\pi n$, then $2x = \frac{5\pi}{3} + 2\pi n$, so $x = \frac{5\pi}{6} + \pi n$.
Therefore, our solutions are: $x = \frac{\pi}{2}\frac{3\pi}{2}\frac{5\pi}{6}\frac{11\pi}{6}$.

(b) $2\cos^2\theta + 5\cos\theta = 3$
If we let $u = \cos\theta$, we then have $2u^2 + 5u = 3$, or $2u^2 + 5u - 3 = 0$,
which factors to give $(2u-1)(u+3) = 0$.
That is, we have $(2\cos\theta - 1) = 0$ or $(\cos\theta + 3) = 0$.
In the first case, $2\cos\theta = 1$, or $\cos\theta = \frac{1}{2}$. Hence $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$.
In the second case, $\cos\theta = -3$, which has no solutions.

(c) $\cos 2t = 1 - 2\sin t$
Using the double angle identity for cos, we have $1 - 2\sin^2 t = 1 - 2\sin t$.
Gathering everything to one side, we have $-2\sin^2 t + 2\sin t = 0$.
Then $-2\sin t(\sin t - 1) = 0$, so either $-2\sin t = 0$, or $\sin t = 1$.
In the first case, $\sin t = 0$, so $t = 0$ or $t = \pi$. In the second case, $\sin t = 1$, so $t = \frac{\pi}{2}$.

5. Given the graph: 

   (a) (3 points) Find $f^{-1}(0)$.
   $f^{-1}(0) = 2$

   (b) (3 points) Find $f^{-1}(3)$.
   $f^{-1}(3) = -1$

   (c) (3 points) Find $f(f^{-1}(2))$.
   $f(f^{-1}(2)) = f(0) = 2$

   (d) (3 points) Find x if $f^{-1}(x) = 2$.
   If $f^{-1}(x) = 2$, then $f(2) = x$, so $x = 0$

6. (6 points) Determine whether or not the function $f(x) = |x|$ is one-to-one.

   Notice that $f(2) = |2| = 2$, while $f(-2) = |-2| = 2$, so this function is **not** one to one.