Math 143
Exam 4 - Solutions **Name:**__________________________________
04/20/2007

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. I will give credit to each problem proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Be sure to simplify answers when possible. Also, make sure to follow directions carefully on each problem.

1. Find the *exact* value of the following:

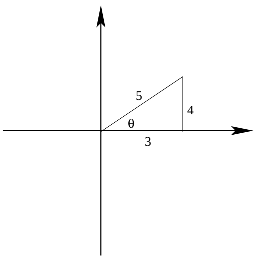
   (a) (4 points) $\cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$

   (b) (4 points) $\sin^{-1}(\frac{3\pi}{2})$ - undefined, since $\frac{3\pi}{2} > 1$

   (c) (5 points) $\cos(\cos^{-1}(\frac{1}{2})) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

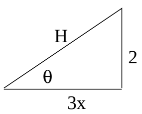
   (d) (5 points) $\sin(\sin^{-1}(-\frac{1}{2})) = \sin(\frac{11\pi}{6}) = -\frac{1}{2}$

   (e) (7 points) $\sin(2\cos^{-1}(\frac{3}{5})) = \sin(2\theta) = 2\sin\theta\cos\theta$, where θ is the angle given by $\cos^{-1}(\frac{3}{5})$.
   Consider the triangle:

   We see that $\sin\theta = \frac{4}{5}$, and $\cos\theta = \frac{3}{5}$, therefore, $\sin(2\cos^{-1}(\frac{3}{5})) = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$.

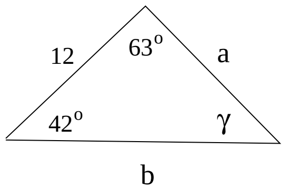
2. (5 points) Express $\sin(\tan^{-1}(\frac{2}{3x}))$ algebraically.
   Consider the related triangle:

   Since $H = \sqrt{2^2 + (3x)^2} = \sqrt{4 + 9x^2}$, $\sin(\tan^{-1}(\frac{2}{3x})) = \sin\theta = \frac{2}{\sqrt{4+9x^2}}$

3. (7 points each) Solve the following triangles: (these are not neccessarily drawn to scale)

(a)



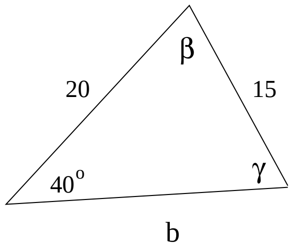
First notice that $\gamma = 180° - 63° - 42° = 75°$

Next, by the Law of Sines, $\dfrac{\sin 42°}{a} = \dfrac{\sin 75°}{12} = \dfrac{\sin 63°}{b}$.

Therefore, $a = \dfrac{12 \sin 42°}{\sin 75°} \approx 8.31$

Similarly, $a = \dfrac{12 \sin 63°}{\sin 75°} \approx 11.07$

(b)



By the Law of Sines, $\dfrac{\sin 40°}{15} = \dfrac{\sin \gamma}{20}$

Therefore, $\sin \gamma = \dfrac{20 \sin 40°}{15} \approx .857$

So either $\gamma = \sin^{-1}(.857) \approx 59.0°$,

or $\gamma \approx 180° - 59° = 121.0°$.

If $\gamma \approx 59.0°$, then $\beta \approx 180° - 59.0° - 40° = 81.0°$,

and $b = \dfrac{15 \sin 81°}{\sin 40°} \approx 23.0$

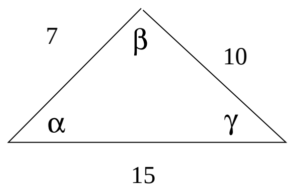
On the other hand, If $\gamma \approx 121°$,

then $\beta \approx 180° - 121.0° - 40° = 19.0°$,

and $b = \dfrac{15 \sin 19°}{\sin 40°} \approx 7.6$

Therefore, we see that there are two possible triangles that meet the given criteria.

(c)



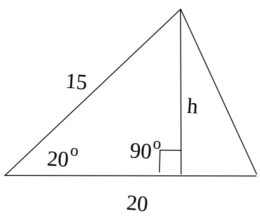
By the Law of Cosines, $a^2 = b^2 + c^2 - 2bc \cos \alpha$, so $\cos \alpha = \dfrac{a^2 - b^2 - c^2}{-2bc}$.

Therefore, $\cos \alpha = \dfrac{(10)^2 - (15)^2 - (7)^2}{-2(15)(7)} \approx .82857$, so $\alpha \approx \cos^{-1}(.82857) \approx 34.0°$

Similarly, Therefore, $\cos \beta = \dfrac{(15)^2 - (10)^2 - (7)^2}{-2(10)(7)} \approx -.54286$, so $\alpha \approx \cos^{-1}(-.54286) \approx 122.9°$

Finally, $\gamma \approx 180° - 34.0° - 122.9° = 23.0°$

4. (5 points) Find the area of the following triangle



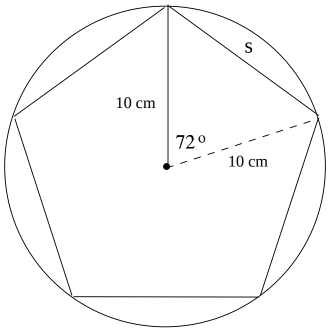
Recall the the area of a triangle is given by $A = \frac{1}{2}bh$.

If we drop a perpendicular to the base of this triangle, we see that $\sin 20° = \frac{h}{15}$, so $h = 15 \sin 20°$.

Therefore, $A = \frac{1}{2}(20)(15 \sin 20°) \approx 51.3$ square units.

Note: One could also just apply either of our area formulas either directly, or after solving the triangle.

5. (7 points) A regular pentagon is inscribed in a circle of radius 10cm. Find the length of one of the sides of the pentagon (See the figure below).



First notice that if we draw in line segments from the center of the circle to two adjacent vertices of the pentagon, we form an isosceles triangle. Since the pentagon is regular, the central angle of this triangle must be $\frac{360°}{5} = 72°$.

Therefore, by the Law of Cosines, $s^2 = 10^2 + 10^2 - 2(10)(10)\cos 72° \approx 138.2$

Hence $s \approx \sqrt{138.2} \approx 11.75$ cm.

6. (4 points each) Express the following in the form $a+bi$. You do **not** have to use trigonometric forms.

   (a) $(5-3i)-(2+7i) = 5-2-3i-7i = 3-10i$

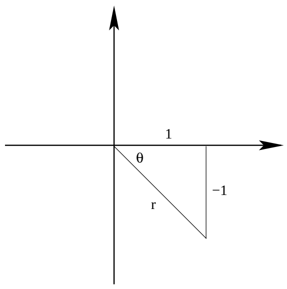
   (b) $(5-3i)(2+7i) = 10-6i+35i-21i^2 = 10+21-6i+35i = 31+29i$

   (c) $\dfrac{5-3i}{2+7i} \cdot \dfrac{2-7i}{2-7i} = \dfrac{10-6i-35i+21i^2}{4-49i^2} = \dfrac{10-21-41i}{4+49} = \dfrac{-11-41i}{53} = -\dfrac{11}{53} - \dfrac{41}{53}$

   (d) $i^{2345} = \left(i^4\right)^{586} \cdot i = i$

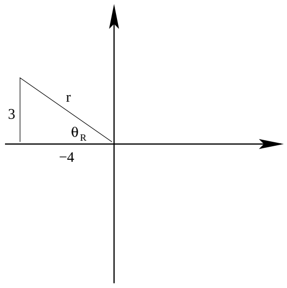
7. Let $z_1 = 1 - i$ and $z_2 = -4 + 3i$

(a) (5 points) Find the trigonometric form of z_1



First, notice that $r_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$.
Next, from the figure, we see that $\theta_1 = -\frac{\pi}{4}$.
Therefore, $z_1 = \sqrt{2} cis \left(-\frac{\pi}{4}\right)$.

(b) (5 points) Find the trigonometric form of z_2



First, notice that $r_2 = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.
Next, from the figure, we see that $\theta_2 = \tan^{-1}\left(-\dfrac{3}{4}\right) + \pi$.
[Note: we have to add π since θ_2 is in the second quadrant]
Therefore, $z_2 = 5cis\left[\tan^{-1}\left(-\dfrac{3}{4}\right) + \pi\right]$

(c) (8 points) Express $(z_1)^5$ in the form $a + bi$
Using De Moivre's Theorem:
$(z_1)^5 = r_1^5 cis\,(5\theta_1) = (\sqrt{2})^5 cis\left(-\frac{5\pi}{4}\right) = 4\sqrt{2} cis\left(\frac{3\pi}{4}\right) = 4\sqrt{2}[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}]$
$= 4\sqrt{2}[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}] = -4 + 4i$.

(d) (8 points) Find the cube roots of $z_1 = 1 - i$
Suppose $w_0 = scis(\alpha_0)$, $w_1 = scis(\alpha_1)$, and $w_2 = scis(\alpha_2)$ are the three distinct cube roots of $z_1 = 1 - i$. Then $s = \sqrt[3]{r_1} = \sqrt[3]{\sqrt{2}} = \sqrt[6]{2}$.
Also, $\alpha_0 = \frac{\theta_1}{3} = \frac{11pi}{12} = \frac{315^\circ}{3} = 105^\circ$
Similarly, $\alpha_1 = \frac{\theta_1}{3} + \frac{2\pi}{3} = 105^\circ + 120^\circ = 225^\circ$
Also, $\alpha_2 = \frac{\theta_1}{3} + \frac{4\pi}{3} = 105^\circ + 240^\circ = 345^\circ$
Therefore, $w_0 = \sqrt[6]{2} cis\,(105^\circ)$, $w_1 = \sqrt[6]{2} cis\,(225^\circ)$, and $w_2 = \sqrt[6]{2} cis\,(345^\circ)$.
(See the figure below)