Math 143

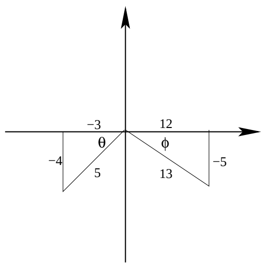
Final Exam Practice Problems - Part 2

1. Find the *exact* value of the following:

   (a) $\sin(\frac{-\pi}{12}) = \sin(-15°) = \sin(30° - 45°)$
   $= \sin(30°)\cos(45°) - \sin(45°)\cos(30°) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$

   (b) $\cos(105°) = \cos(60° + 45°)$
   $= \cos(60°)\cos(45°) - \sin(60°)\sin(45°) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$

2. Given that $\csc\theta = -\frac{5}{4}$ and $\cos\phi = \frac{12}{13}$, where θ is in the third quadrant, and ϕ is in the fourth quadrant, find *exact* values of:



   (a) $\cos\theta = -\frac{3}{5}$

   (b) $\sin(2\phi) = 2\sin\phi\cos\phi = 2\left(-\frac{5}{13}\right)\left(\frac{12}{13}\right) = -\frac{120}{169}$

   (c) $\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta = \left(\frac{12}{13}\right)\left(-\frac{3}{5}\right) + \left(-\frac{5}{13}\right)\left(-\frac{4}{5}\right) = \frac{-36+20}{65} = -\frac{16}{65}$

   (d) $\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta = \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{5}{13}\right)\left(-\frac{3}{5}\right) = \frac{-48+15}{65} = -\frac{33}{65}$

3. Verify the following identities by transforming the left hand side into the right hand side:

   (a) $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$
   $$\cos\left(\frac{3\pi}{2} - \theta\right) = \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta = 0\cdot\cos\theta + (-1)\sin\theta = -\sin\theta$$

   (b) $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} = 2\tan x\sec x$
   $$\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{1+\sin x - (1-\sin x)}{1-\sin^2 x} = \frac{1+\sin x - 1 + \sin x}{\cos^2 x}$$
   $= \frac{2\sin x}{\cos^2 x} = 2\tan x\sec x.$

   (c) $\sec^2 t - \csc^2 t = \dfrac{\tan t - \cot t}{\sin t\cos t}$
   $$\sec^2 t - \csc^2 t = \frac{1}{\cos^t} - \frac{1}{\sin^2 t} = \frac{\sin^2 t - \cos^2 t}{\cos^2 t\sin^2 t} = \frac{\sin^2 t - \cos^2 t}{(\cos t\sin t)^2} = \frac{\frac{\sin^2 t - \cos^2 t}{\cos t\sin t}}{\cos t\sin t}$$
   $$= \frac{\frac{\sin^2 t}{\cos t\sin t} - \frac{\cos^2 t}{\cos t\sin t}}{\cos t\sin t} = \frac{\frac{\sin t}{\cos t} - \frac{\cos t}{\sin t}}{\cos t\sin t} = \frac{\tan t - \cot t}{\cos t\sin t}$$

   (d) $\sin 3t\cos 3t = \dfrac{1}{2}\sin 6t$
   $\sin 3t\cos 3t = \frac{1}{2}(2\sin 3t\cos 3t) = \frac{1}{2}\sin 6t.$

   (e) $\dfrac{1+\cos 2t}{\sin 2t} = \cot t$
$$\frac{1+\cos 2t}{\sin 2t} = \frac{1+\cos^2 t - \sin^2 t}{2\sin t\cos t} = \frac{1-\sin^2 t + \cos^2 t}{2\sin t\cos t} = \frac{1+\cos^2 t + \cos^2 t}{2\sin t\cos t} = \frac{2\cos^2 t}{2\sin t\cos t} = \frac{\cos t}{\sin t} = \cot t.$$

4. Find *exact* solutions to the following equations with $0 \leq \theta < 2\pi$.

(a) $2\sin(4x + \frac{\pi}{4}) = -\sqrt{3}$
Then $\sin(4x + \frac{\pi}{4}) = -\frac{\sqrt{3}}{2}$, or $\sin\theta = -\frac{\sqrt{3}}{2}$, where $\theta = 4x + \frac{\pi}{4}$
Therefore, $4x + \frac{\pi}{4} = \frac{4\pi}{3} + 2\pi k$ or $4x + \frac{\pi}{4} = \frac{5\pi}{3} + 2\pi k$
But then, $4x = \frac{13\pi}{12} + 2\pi k$ or $4x = \frac{17\pi}{12} + 2\pi k$
Thus $x = \frac{13\pi}{48} + \frac{\pi}{2}k$ or $x = \frac{17\pi}{48} + \frac{\pi}{2}k$
Hence $x = \frac{13\pi}{48}, \frac{37\pi}{48}, \frac{61\pi}{48}, \frac{85\pi}{48}$ or $x = \frac{17\pi}{48}, \frac{41\pi}{48}, \frac{65\pi}{48}, \frac{89\pi}{48}$.

(b) $4\cos^3\theta = 3\cos\theta$
Then $4\cos^3\theta - 3\cos\theta = 0$, or $\cos\theta(4\cos^2\theta - 3) = 0$
Thus either $\cos\theta = 0$, or $4\cos^2\theta - 3 = 0$, so $4\cos^2\theta = 3$, or $\cos^2\theta = \frac{3}{4}$, so $\cos\theta = \pm\frac{\sqrt{3}}{2}$.
Hence $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$

(c) $\sin 2t - \sin t = 0$
Then $2\sin t\cos t - \sin t = 0$, so $\sin t(2\cos t - 1) = 0$. Thus either $\sin t = 0$, or $\cos t = \frac{1}{2}$.
Hence $t = 0, \pi, \frac{\pi}{3}$, and $\frac{5\pi}{3}$.

(d) $2\cos^2\theta - 5\cos\theta - 5 = 0$
Notice this equation is quadratic in form. Therefore, we let $u = \cos t$.
We then have $2u^2 - 5u - 5 = 0$.
Solving this using the quadratic formula: $u = \frac{5\pm\sqrt{25-4(2)(-5)}}{2(2)} = \frac{5}{4} \pm \frac{\sqrt{65}}{4}$.
Therefore, $\cos\theta = \frac{5}{4} \pm \frac{\sqrt{65}}{4}$. So either $\cos\theta \approx 3.26556$, or $\cos\theta = -.76556$
Then $\theta = \cos^{-1}(-.76556) = 2.443$ radians, or $\theta = 2\pi - 2.443 = 3.840$ radians.

5. Given the tables below, find the following:

x	0	2	4	6	8
f(x)	1	5	8	4	0

x	0	2	4	6	8
g(x)	2	6	5	9	7

(a) $f^{-1}(5) = 2$
(b) $f(g^{-1}(9)) = f(6) = 4$
(c) $g(f^{-1}(4)) = g(6) = 9$

6. Determine whether or not the following functions are one-to-one. You must justify your answer to each part.

(a) $f(x) = 3x^2 - 2$
$f(x)$ is not one-to-one. Notice that $f(1) = f(-1) = 1$.

(b) $g(x) = \dfrac{4}{x}$
Suppose that $g(x_1) = g(x_2)$ for two x-values x_1 and x_2.
Then $\frac{4}{x_1} = \frac{4}{x_2}$, but then $4x_2 = 4x_1$, or $x_2 = x_1$.
Hence $g(x)$ is a one-to-one function.

7. Find the *exact* value of the following:

   (a) $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$

   (b) $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

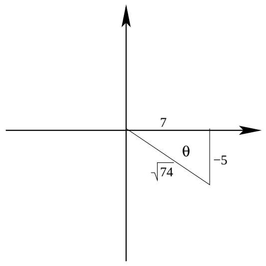
   (c) $\cos^{-1}(-\pi)$ is undefined.

   (d) $\cos(\cos^{-1}(-\frac{1}{2})) = \cos(\frac{2\pi}{3}) - \frac{1}{2}$

   (e) $\sin^{-1}(\sin(\frac{2\pi}{3}) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$

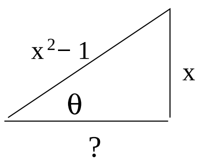
   (f) $\tan(\cos^{-1}(\frac{1}{2})) = \tan(\frac{\pi}{3}) = \sqrt{3}$

   (g) $\cos(2\tan^{-1}(-\frac{5}{7})) = \cos(2\theta) = \cos^2\theta - \sin^2\theta$, where θ is described below:



   Then $\cos^2\theta - \sin^2\theta = \left(\frac{7}{\sqrt{74}}\right)^2 - \left(\frac{-5}{\sqrt{74}}\right)^2 = \frac{49-25}{74} = \frac{24}{74}$

8. Express $\tan(\cos^{-1}(\frac{x}{x^2-1}))$ algebraically.

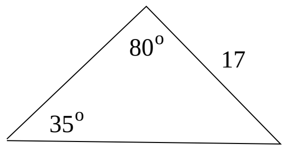


   First, we solve for the missing side in the triangle above using the Pythagorean Theorem in order to obtain: $\sqrt{(x^2-1)^2 - x^2} = \sqrt{x^4 - 2x^2 - 1 - x^2} = \sqrt{x^4 - 3x^2 - 1}$.

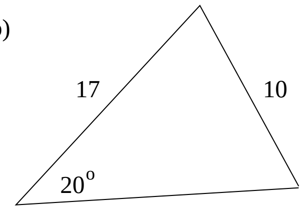
   Thus $\tan(\cos^{-1}(\frac{x}{x^2-1})) = \frac{x}{\sqrt{x^4-3x^2-1}}$.

9. Solve the following triangles: (these are not neccessarily drawn to scale)

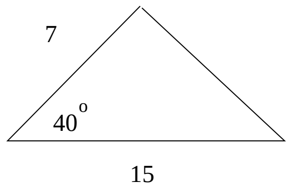
(a)



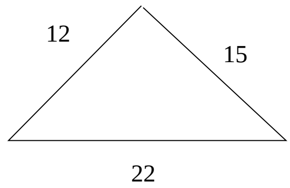
(b)



(c)



(d)



For triangle (a), $\gamma = 180° - 80° - 35° = 65°$, $b = \frac{17 \sin 80°}{\sin 35°} \approx 29.2$, and $c = \frac{17 \sin 65°}{\sin 35°} \approx 26.9$

For triangle (b), $\sin \gamma = \frac{17 \sin 20°}{10} \approx .5814$. Therfore, either $\gamma = 35.6°$, or $\gamma = 180° - 35.6° = 144.4°$.

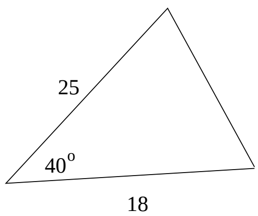
If $\gamma = 35.6°$, then $\beta = 124.4°$, and $b = \frac{10 \sin 124.4°}{\sin 20°} \approx 24.12$.

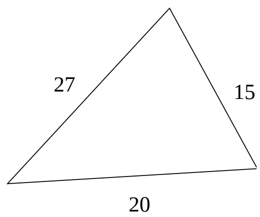
If $\gamma = 124.4°$, then $\beta = 15.6°$, and $b = \frac{10 \sin 15.6°}{\sin 20°} \approx 7.86$.

For triangle (c), we have that $a^2 = 15^2 + 7^2 - 2(15)(7)\cos 40° \approx 113.1307$, So $a \approx 10.6$. Then $\cos \beta = \frac{15^2 - 7^2 - 10.6^2}{-2(7)(10.6)} \approx -.4288$, and $\beta \approx 115.4°$. Thus $\gamma = 180° - 40° - 115.4° = 24.6°$.

For triangle (d), we have that $\cos \alpha = \frac{15^2 - 12^2 - 22^2}{-2(12)(22)} \approx .76326$, and so $\alpha \approx 40.2°$. Similarly, $\beta = \frac{22^2 - 12^2 - 15^2}{-2(12)(15)} \approx -.31944$, and so $\alpha \approx 108.6°$.Thus $\gamma = 180° - 108.6° - 40.2° = 31.2°$.

10. Find the area of the following triangles



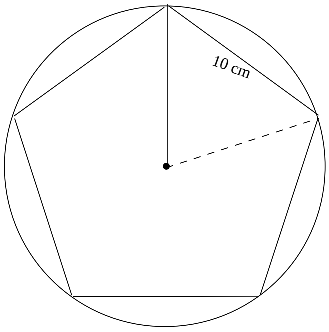


For the rightmost triangle, if we drop a perpendiclar to the base, we have $\frac{h}{25} = \sin 40°$, or $h = 25 \sin 40°$. Then since $A = \frac{1}{2}bh = \frac{1}{2}(18)(25)\sin 40° \approx 144.63$ square units.

For the other triamgle, we use Heron's formula.

$s = \frac{1}{2}(27 + 15 + 20) = 31$, so $A = \sqrt{(31)(31-27)(31-20)(31-15)} = \sqrt{21824} \approx 147.73$

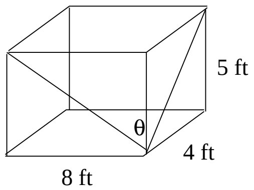
11. Suppose that a regular pentagon inscribed in a circle has sides of length 10cm. Find the area of the pentagon (See the figure below).



First notice that we can find the area of the pentagon by subdividing it into 5 congruent triangles and then finding the area of one of these triangles. Recall that the central angle of the triangle is $\frac{360}{5} = 72°$, and the triangle is isosceles, so the base angles are each $54°$. Therefore, the base of the triangle has length 10cm, and the other two sides have length $x = \frac{10 \sin 54°}{\sin 72°} \approx 8.5$cm. Then the height of the triangle is given by $h \approx 8.5 \sin 54°$. Hence the area of the triangle is given by $A = \frac{1}{2}(10)(8.5) \sin 54° \approx 34.38 cm^2$.

Thus, the area of the pentagon is approximately $5(34.38) = 171.92 cm^2$.

12. A rectangular box measures 8 feet by 4 feet by 5 feet. Find the angle θ between the diagonal on the front of the box with the diagonal on one of the sides of the box. (See the figure below).



We use the Pythagorean Theorem in order to find the lengths of the diagonals on the front, top, and side of the box:

$d_f = \sqrt{8^2 + 5^2} = \sqrt{64 + 25} = \sqrt{89}$, $d_s = \sqrt{4^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$, and $d_t = \sqrt{8^2 + 4^2} = \sqrt{64 + 16} = \sqrt{80}$.

Next, applying the Law of Cosines to the triangle formed by the three diagonals:

$\cos\theta = \frac{80 - 89 - 41}{-2(\sqrt{89})(\sqrt{41})} \approx .41386$, so $\theta \approx 65.6°$

13. Express the following in the form $a + bi$. You do **not** have to use trigonometric forms.

(a) $(7 - 2i) - (6 + 11i) = 1 - 3i$

(b) $(7 - 2i)(6 + 11i) = 42 - 12i + 77i - 22i^2 = 64 + 65i$

(c) $\frac{7 - 2i}{6 + 11i} \cdot \frac{6 - 11i}{6 - 11i} = \frac{42 - 12i - 77i + 22i^2}{36 - 121i^2} = \frac{20 - 89i}{157} = \frac{20}{157} - \frac{89}{157}i$

(d) $i^{23456} = (i^4)^{5864} = 1^{5864} = 1$

14. Let $z_1 = -5 - 5i$ and $z_2 = -12 + 5i$

(a) Find the trigonometric form of z_1
$r = \sqrt{5^2 + 5^2} = 5\sqrt{2}$. θ is best found by plotting z_1 and using common sense.
$z_1 = 5\sqrt{2} cis \left(\frac{5\pi}{4}\right)$

(b) Find the trigonometric form of z_2
$r = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$.
$\theta = \tan^{-1}\left(-\frac{5}{12}\right) + \pi$
So $z_2 = 13 cis \left(\tan^{-1}\left(-\frac{5}{12}\right) + \pi\right)$.

(c) Express $(z_1)^6$ in the form $a + bi$
$z_1^6 = \left(\sqrt{50}\right)^6 cis \left(6 \cdot \frac{5\pi}{4}\right) = 50^3 cis \left(\frac{30\pi}{4}\right) = 125,000 cis \left(\frac{3\pi}{2}\right) = 125,000(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) = -125,000i$.

(d) Find the fourth roots of $z_1 = -5 - 5i$
$s = \sqrt[4]{r} = \sqrt[4]{\sqrt{50}} = \sqrt[8]{50}$
$\alpha_0 = \frac{\theta}{4} = \frac{225}{4} = 56.25°$
Thus $w_0 = \sqrt[8]{50} cis(56.25°)$, $w_1 = \sqrt[8]{50} cis(56.25° + 90°) = \sqrt[8]{50} cis(146.25°)$, $w_3 = \sqrt[8]{50} cis(56.25° + 180°) = \sqrt[8]{50} cis(236.25°)$, and $w_4 = \sqrt[8]{50} cis(56.25° + 270°) = \sqrt[8]{50} cis(326.25°)$.

Note: Since the material on polar coordinates from Chapter 11 is not required material for the final exam, I am going to omit the assigned problems from this material on the review assignment.