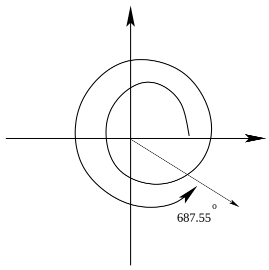
Math 143

Final Exam Practice Problems

1. Given the angle $\theta = 12$ radians

   (a) Express θ in terms of degrees, with your answer rounded to the nearest hundredth of a degree. $12rad \cdot \frac{180°}{\pi \text{rad}} \approx 687.55°$

   (b) Draw θ in standard position



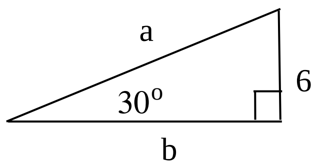
   (c) Convert θ into degree, minute, second form. - $687.55° = 687°33'0''$

   (d) Find one positive angle and one negative angle (in degrees) that are coterminal with θ
   There are many answers to this question, for example, $327.55°$ and $-32.45°$

2. You are aerating a rectangular lawn that is 40 feet long and 60 feet wide using an aerator whose cylindrical drum is 4 feet wide, and has a radius of 15 inches. If you complete job in the most efficient way possible, how many revolutions will the aerator drum have rotated from the beginning to the end of the job?

   Notice that if we aerate by rolling the aerator parallel with the 60 ft side, we will have to make 10 passes with the aerator, so we will have rolled it a total of $(60)(10) = 600$ feet. Next, the circumferance of the drum on the aerator is: $C = 2\pi r = 2\pi(1.25\text{ft}) \approx 7.854$, so the drum covers about 7.854 feet of linear distance each revolution. Thus the drum of the aerator will have completed $\frac{600}{7.853} \approx 76.4$ revolutions.

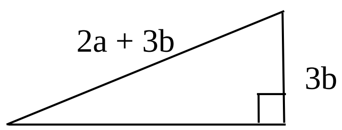
3. Find the values of a and b *exactly* based on the triangle below:



   Since $\sin 30° = \frac{6}{a}$, $a = \frac{6}{\sin 30°} = \frac{6}{.5} = 12$

   similarly, $b = 12\cos 30° = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$

4. Find algebraic expressions for the 6 basic trig functions based on the triangle in the figure below:



   First, we use the Pythagorean Theorem to find the missing side of the triangle:

   $x^2 = (2a+3b)^2 - (3b)^2 = 4a^2 + 12ab + 9b^2 - 9b^2 = 4a^2 + 12ab = 4(a^2 + 3ab)$

   So $x = 2\sqrt{a^2 + 3ab}$. Therefore, using the fundamental identities:

   $\sin\theta = \frac{3b}{2a+3b}$, $\cos\theta = \frac{2\sqrt{a^2+3ab}}{2a+3b}$, $\tan\theta = \frac{3b}{2\sqrt{a^2+3ab}}$

   $\csc\theta = \frac{2a+3b}{3b}$, $\sec\theta = \frac{2a+3b}{2\sqrt{a^2+3ab}}$, $\cot\theta = \frac{2\sqrt{a^2+3ab}}{3b}$

5. Use trigonometric identities to write $\sec\theta$ as a function of $\sin\theta$.

First, recall that $\sec\theta = \frac{1}{\cos\theta}$. Next, by the Pythagorean identity, $sin^2\theta + \cos^2\theta = 1$, so $\cos^2\theta = 1 - \sin^2\theta$, and thus $\cos\theta = \sqrt{1-\sin^2\theta}$. Therefore, $\sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}}$

6. Verify the following identities by transforming the left hand side into the right hand side:

(a) $\tan x \csc x \cos x = 1$

$\tan x \csc x \cos x = \frac{\sin x}{\cos x}\frac{1}{\sin x}\cos x = \frac{\sin x \cos x}{\cos x \sin x} = 1.$

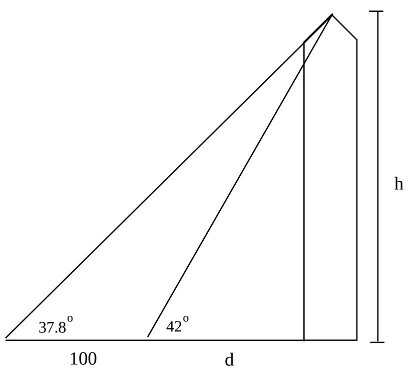
(b) $\frac{\sin\theta}{1-\cos\theta} = \csc\theta + \cot\theta$

$\frac{\sin\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{\sin\theta+\sin\theta\cos\theta}{1-\cos^2\theta} = \frac{\sin\theta+\sin\theta\cos\theta}{\sin^2\theta} = \frac{\sin\theta}{\sin^2\theta} + \frac{\sin\theta\cos\theta}{\sin^2\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \csc\theta + \cot\theta.$

7. Fill in *exact* values in each blank in the table below:

θ (degrees)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undef.
$180°$	π	0	-1	0
$270°$	$\frac{3\pi}{2}$	-1	0	undef.

8. From a point A on a line from the base of the Washington Monument, the angle of elevation to the top of the momument is $42.0°$. From a point B 100 feet further away along the same line, the angle to the top is $37.8°$. Find the height of the momument to the nearest foot.



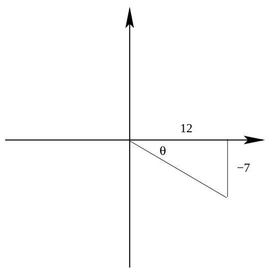
Let h be the height of the monument, and d the distance from the closer of the two points to the base of the monument. Then, from the figure above, we see that $\tan 42° = \frac{h}{d}$, and $\tan 37.8° = \frac{h}{100+d}$, or $h = d\tan 42° = (d+100)\tan 37.8°$.

Then $d\tan 42° = d\tan 37.8° + 100\tan 37.8°$, or $d\tan 42° - d\tan 37.8° = 100\tan 37.8°$. That is, $d(\tan 42° - \tan 37.8°) = 100\tan 37.8°$. Thus $d = \frac{100\tan 37.8°}{\tan 42° - \tan 37.8°} \approx 621.91$ feet.

From this, $h = d\tan 42° \approx (621.91)\tan 42° \approx 560$ feet.

9. Given that $\tan\theta = -\frac{7}{12}$ and $\sin\theta < 0$, find the exact value of both $\cos\theta$ and $\csc\theta$.

   Since $\tan\theta$ and $\sin\theta$ are both negative, θ must be in the fourth quadrant. Therefore, we can look at the following triangle:

   Based on this, we use the Pythagorean theroem to find the hypotenuse: $h = \sqrt{144+49} = \sqrt{193}$

   From this, $\cos\theta = \frac{12}{\sqrt{193}}$, and $\sec\theta = -\frac{\sqrt{193}}{7}$

10. Find the *exact* value of the following:

    (a) $\sin(-135°) = \sin(225°) = -\sin(45°) = \frac{\sqrt{2}}{2}$

    (b) $\tan(\frac{5\pi}{3}) = -\tan\frac{\pi}{3} = -\sqrt{3}$

    (c) $\cot(540°) = \cot(180°) = \frac{1}{\tan(180°)} = \frac{1}{0}$ - undefined.

    (d) $\cos(\frac{5\pi}{12}) = \cos(75°) = \cos(45°+30°) = \cos(45°)\cos(30°) - \sin(45°)\sin(30°) = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

11. Find all solutions to the following equations with $0 \le \theta \le 360°$. Find exact answers whenever possible. Otherwise, approximate your solutions to within .01°

    (a) $\tan\theta = -12$

    Then $\theta = \tan^{-1}(-12) + 180° \cdot k \approx -85.24°180° \cdot k$, so our solutions between 0° and 360° are: $\theta = 94.76°$ and $\theta = 274.76°$

    (b) $\cos 2\theta = -\frac{1}{2}$

    Notice that if $\cos x = -\frac{1}{2}$, then either $x = \frac{2\pi}{3} + 2\pi k$ or $x = \frac{4\pi}{3} + 2\pi k$. That is, either $2\theta = \frac{2\pi}{3} + 2\pi k$ or $2\theta = \frac{4\pi}{3} + 2\pi k$.

    Solving these, either $\theta = \frac{\pi}{3} + \pi k$ or $\theta = \frac{2\pi}{3} + \pi k$.

    Therefore, $\theta = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$, which in degrees is $\theta = 60°, 240°, 120°$, or $300°$.

    (c) $3\sin(4\theta - \pi) = -\frac{3\sqrt{3}}{2}$

    Then $\sin(4\theta - \pi) = -\frac{\sqrt{3}}{2}$. Notice that if $\sin x = -\frac{\sqrt{3}}{2}$, then $x = \frac{4\pi}{3} + 2\pi k$ or $x = \frac{5\pi}{3} + 2\pi k$.

    That is, we have $4\theta - \pi = \frac{4\pi}{3} + 2\pi k$ or $4\theta - \pi = \frac{5\pi}{3} + 2\pi k$

    Then $4\theta = \frac{7\pi}{3} + 2\pi k$ or $4\theta = \frac{8\pi}{3} + 2\pi k$

    Hence $\theta = \frac{7\pi}{12} + \frac{\pi}{2}k$ or $\theta = \frac{8\pi}{12} + \frac{\pi}{2}k$

    Translating this into degrees, $\theta = 105°, 195°, 285°$, and $15°$, or $\theta = 120°, 210°, 300°$, and $30°$ are our solutions between 0° and 360°

12. Determine whether the following function are even, odd, or neither:

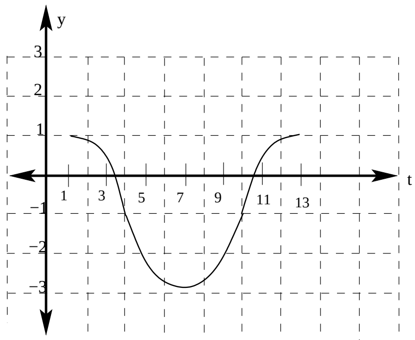
    (a) $f(x) = x - 2\sin x$

    $f(-x) = -x - 2\sin(-x) = -x + 2\sin(x)$, so $f(x)$ is odd.

    (b) $g(x) = 4\cos x + x^3$

    $g(-x) = 4\cos(-x) + (-x)^3 = 4\cos x - x^3$, so $g(x)$ is neither even nor odd.

13. Given the graph:



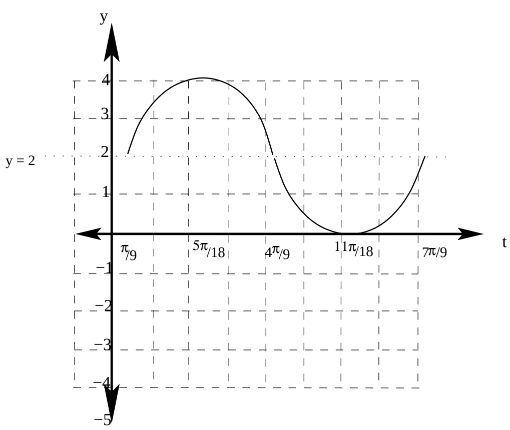
   (a) Find the amplitude, period, and midline for the graph.
   amplitude: 2; midline: $y = -1$; period: 12 (so $b = \frac{2\pi}{12} = \frac{\pi}{6}$

   (b) Express the function shown with an equation of the form: $y = a\sin(bt + c) + d$
   $y = -2\sin\left(\frac{\pi}{6}t - \frac{2\pi}{3}\right)$

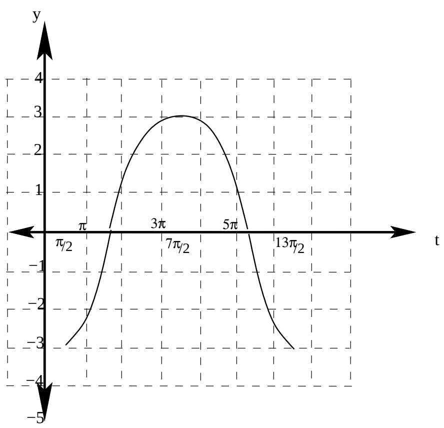
   (c) Express the function shown with an equation of the form: $y = a\cos(bt + c) + d$
   $y = 2\cos\left(\frac{\pi}{6}t - \frac{\pi}{6}\right)$

14. For each function below, find the amplitude and period of the function, and then carefully draw the graph the the function.

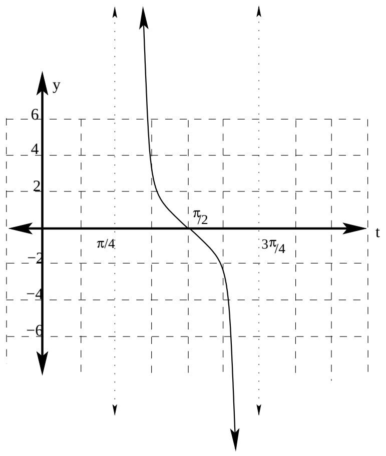
   (a) (6 points) $y = 2\sin(3t - \frac{\pi}{3}) + 2$
   amplitude: 2; period: $\frac{2\pi}{3}$



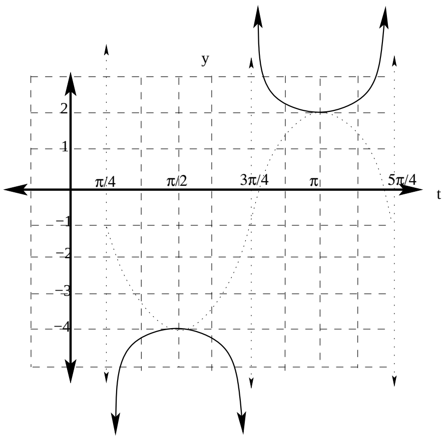
   (b) (6 points) $y = -3\cos(\frac{1}{3}t - \frac{\pi}{6})$
   amplitude: 3; period: $\frac{2\pi}{\frac{1}{3}} = 6\pi$



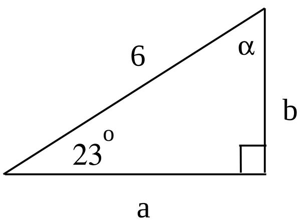
(c) (6 points) $y = 4\tan(2t - \pi)$
amplitude: 4; period: $\frac{\pi}{2}$

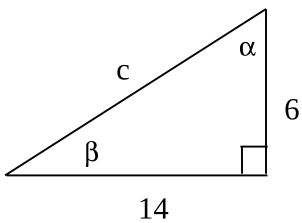


(d) (6 points) $y = -3\csc(2t - \frac{\pi}{2}) - 1$
amplitude: 3; period: π



15. Given the indicated parts of the triangle $\triangle ABC$, find all remaining parts. Estimate your answers to within 2 decimal places.

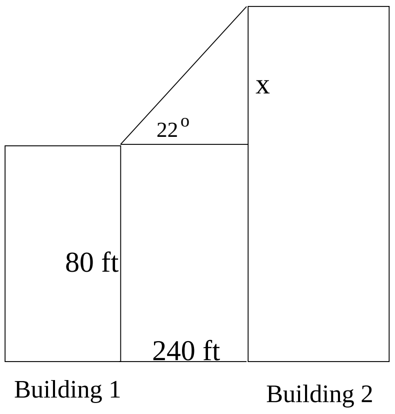




(a) $\sin 23° = \frac{b}{6}$, so $6\sin 30° = b$ and $b \approx 2.34$. Similarly, $a = 6\cos 23°$ or $a \approx 5.52$. Finally, $\alpha = 90° - 23° = 67°$.

(b) $c = \sqrt{14^2 + 6^2} = \sqrt{180} = 12\sqrt{5}$. $\sin\beta = \frac{6}{12\sqrt{5}} \approx .4472$, so $\beta \approx 26.56°$, and $\alpha = 90° - 26.56° = 63.44°$.

16. Two buildings are 240 feet apart. The angle of elevation from the top of the shorter building to the top of the taller building is $22°$. It the shorter building is 80 feet high, how high is the taller building.



Notice that $\tan 22° = \frac{x}{240}$, so $x = 240 \tan 22° \approx 96.97$.

Therefore, the height of the taller building is approximately 177 feet.