Math 229
Exam 2 - Solutions **Name:**______________________________
02/21/2007

**Instructions:** You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. I will give credit to each problem proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Be sure to simplify answers when possible. Also, make sure to follow directions carefully on each problem.

1. Evaluate the following limits. Be sure to show enough work to justify your answers.

   (a) (4 points) $\lim\limits_{x\to 0} \dfrac{x^2-2x}{2x^2-x-6}$

   **Solution:**

   $$\lim_{x\to 0} \frac{x^2-2x}{2x^2-x-6} = \frac{0^2-2(0)}{2(0)^2-0-6} = \frac{0}{-6} = 0$$

   (b) (4 points) $\lim\limits_{x\to 2} \dfrac{x^2-2x}{2x^2-x-6}$

   **Solution:**

   First notice that this limit cannot be evaluated directly, since it leads to the indeterminate form $\frac{0}{0}$. Therefore, we attempt to salvage things by simplifying using algebra:

   $$\lim_{x\to 2} \frac{x^2-2x}{2x^2-x-6} = \lim_{x\to 2} \frac{x(x-2)}{(2x+3)(x-2)} = \lim_{x\to 2} \frac{x}{2x+3} = \frac{2}{2(2)+3} = \frac{2}{7}$$

   (c) (4 points) $\lim\limits_{x\to 2} \dfrac{x}{x-2}$

   **Solution:**

   First notice that this limit cannot be evaluated directly, since it leads to the form $\frac{0}{-2}$, which leads us to suspect that this limit might not exist. To verify this, we investigate by evaluating the expression inside the limit at points close to 2.

   | $x$ | 1.9 | 2.1 | 1.99 | 2.01 | 1.999 | 2.001 |
   |---|---|---|---|---|---|---|
   | $f(x)$ | -19 | 21 | -199 | 201 | -1999 | 2001 |

   Since the values are diverging as we get closer and closer to 2, we can conclude that the limit does not exist.

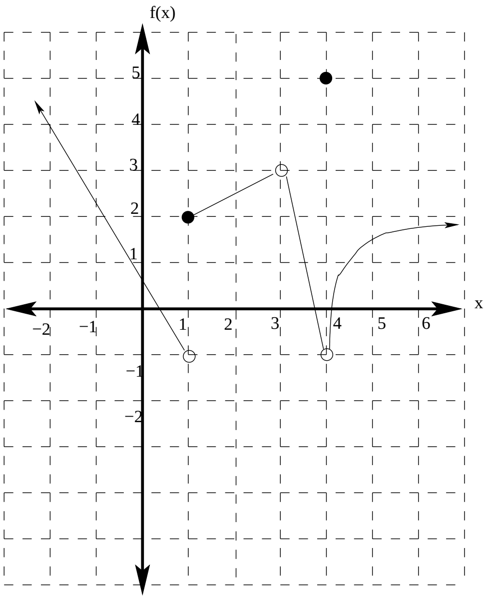
   (d) (4 points) $\lim\limits_{x\to \infty} \dfrac{x^2-2x}{2x^2-x-6}$

   **Solution:**

   For this limit, since we are looking at the limit as it approaches positive infinity, we need only conside the highest order terms:

   $$\lim_{x\to \infty} \frac{x^2-2x}{2x^2-x-6} = \lim_{x\to \infty} \frac{x^2}{2x^2} = \frac{1}{2}$$

2. Given the following graph:




(a) (3 points) Find $\lim_{x \to 1^-} f(x) = -1$

(b) (3 points) Find $\lim_{x \to 1^+} f(x) = 2$

(c) (3 points) Find $\lim_{x \to 4} f(x) = -1$

(d) (3 points) Find $\lim_{x \to \infty} f(x) = 2$

(e) (5 points) List all points where $f(x)$ is discontinuous. Explain what goes wrong at each point.
Notice that $f(x)$ is discontinuous at $x = 1, x = 3$, and $x - 4$, and is continuous everywhere else.
At $x = 1$, $f(x)$ is discontinuous since $\lim_{x \to 1} f(x)$ does not exist.
At $x = 3$, $f(x)$ is discontinuous since $f(3)$ is undefined.
At $x = 4$, $f(x)$ is discontinuous since $\lim_{x \to 4} f(x) = -1$, while $f(4) = 5$, so the limit and the function value do not agree.

3. (10 points) Use the limit definition of the derivative to compute the derivative function $f'(x)$ if $f(x) = 4 - 2x - 3x^2$

**Solution:**

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4-2(x+h)-3(x+h)^2-(4-2x-3x^2)}{h}$$

$$= \lim_{h\to 0} \frac{4-2x-2h-3(x^2+2xh+h^2)-4+2x+3x^2}{h}$$

$$= \lim_{h\to 0} \frac{4-2x-2h-3x^2-6xh-3h^2-4+2x+3x^2}{h} = \lim_{h\to 0} \frac{-2h-6xh-3h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-2-6x-3h)}{h} = \lim_{h\to 0} -2-6x-3h = -2-6x.$$

4. Suppose $f(x) = x^3 - 3x^2 + 5$.

   (a) (5 points) Find the equation for the tangent line to $f(x)$ when $x = 1$.

   **Solution:**

   First, we find the point of tangency by evaluating $f(x)$ when $x = 1$:
   $f(1) = 1^3 - 3(1)^2 + 5 = 1 - 3 + 5 = 3.$
   Next, we find the slope of the tangent line by evaluating the derivative of $f(x)$ when $x = 1$:
   $f'(x) = 3x^2 - 6x$, so $f'(1) = 3 - 6 = -3.$
   Finally, we use the point slope formula to find the equation for the line:
   $y - 3 = -3(x-1) = -3x + 3.$ so $y = -3x + 6.$

   (b) (5 points) Find the value(s) of $x$ for which the tangent line to $f(x)$ is horizontal.

   **Solution:**

   Recall that the tangent line to a function is horizontal if and only if slope of the tangent line is zero, that is, when the derivative of the function is zero.
   Therefore, we consider the equation $f'(x) = 3x^2 - 6x = 0$, or $3x(x-2) = 0$. This equation has two solutions: $x = 0$, and $x = 2$.

5. Find the derivative of each of the following functions. You **do not** have to use the limit definition, and you **do not** need to simplify your answers.

(a) (6 points) $h(x) = x^3 + \sqrt{x^3}$

**Solution:**

First, we rewrite $h(x) = x^3 + x^{\frac{3}{2}}$. Therefore, $f'(x) = 3x^2 + \frac{3}{2}x^{\frac{1}{2}}$

(b) (6 points) $h(x) = \dfrac{5x^3 - 4x^2 + 7x}{x^2}$

Here, we again rewrite $h(x)$ in order to obtain $h(x) = \frac{5x^3}{x^2} - \frac{4x^2}{x^2} + \frac{7x}{x^2} = 5x - 4 + 7x^{-1}$.
Therefore, $h'(x) = 5 - 7x^{-2}$.

(c) (6 points) $h(x) = (x^2 - 4x^3)(4x^3 + 3x^2 - 7x + 3)$

**Solution:**

Using the product rule: $h'(x) = f'(x)g(x) + f(x)g'(x)$,
$h'(x) = (2x - 12x^2)(4x^3 + 3x^2 - 7x + 3) + (x^2 - 4x^3)(12x^2 + 6x - 7)$.

(d) (6 points) $h(x) = (x^3 - 2x + 1)^{\frac{5}{2}}$

**Solution:**

Using the Chain rule: $h'(x) = g'(f(x))f'(x)$, $h'(x) = \frac{5}{2}(x^3 - 2x + 1)^{\frac{3}{2}}(3x^2 - 2)$

(e) (6 points) $\left(\dfrac{2 - 4x^3}{x^2 - 1}\right)^4$

**Solution:**

This derivative requires both the Chain rule and the quotient rule. If we think of $h(x) = g(f(x))$, where $g(x) = x^4$, and $f(x) = \frac{2-4x^3}{x^2-1}$, then since $f'(x) = \frac{(-12x^2)(x^2-1)-(2-4x^3)(2x)}{(x^2-1)^2}$, we see that

$$h'(x) = 4\left(\frac{2 - 4x^3}{x^2 - 1}\right)^3 \cdot \frac{(-12x^2)(x^2 - 1) - (2 - 4x^3)(2x)}{(x^2 - 1)^2}$$

6. Suppose you own a company that manufactures widgets, and the demand equation for them is given by $3x + 4p = 120$.

(a) (5 points) Find the revenue function $R(x)$, and use it to compute $R(10)$ and $R(40)$.

**Solution:**

To find $R(x)$, we solve the demand equation for $p$, yielding $4p = 120 - 3x$, or $p = 30 - \frac{3}{4}x$.
Since revenue is price times quantity, $R(x) = (30 - \frac{3}{4}x)x = 30x - \frac{3}{4}x^2$.
Therefore, $R(10) = 30(10) - \left(\frac{3}{4}\right)(10)^2 = 300 - \frac{300}{4} = 300 - 75 = \$225$.
Similarly, $R(40) = 30(40) - \left(\frac{3}{4}\right)(40)^2 = 1200 - 1200 = \$0$.

(b) (4 points) Find the marginal revenue function $R'(x)$

**Solution:**

$R'(x) = 30 - \frac{3}{2}x$.

(c) (4 points) Compute $R'(10)$ and $R'(40)$ and explain what these numbers mean in practical terms.

**Solution:**

$R'(10) = 30 - \left(\frac{3}{2}\right)(10) = 30 - 15 = 15$. This means that when 10 units have been sold, revenue is changing at \$ 15 per widget, that is, if an additional widget were sold, revenue would increase by about \$ 15.
$R'(40) = 30 - \left(\frac{3}{2}40\right) = 30 - 60 = -30$. This means that when 40 units have been sold, revenue is changing at \$ 0 per widget, that is, revenue is not changing at this point.

(d) (5 points) If $C(x) = 20x + \frac{1}{4}x^2 + 100$, find $P(x)$ and use it to compute $P(10)$.

**Solution:**

Recall that $P(x) = R(x) - C(x) = 30x - \frac{3}{4}x^2 - (20x + \frac{1}{4}x^2 + 100) = 10x - x^2 - 100$.
Therefore, $P(10) = 10(10) - 10^2 - 100 = 100 - 100 - 100 = -100$.

(e) (4 points) Find the marginal profit function $P'(x)$, use it to compute $P'(5)$, and explain what this means in practical terms.

**Solution:**

$P'(x) = 10 - 2x$
$P'(5) = 10 - 2(5) = 0$. In practical terms, this means that when 5 widgets have been sold, profit is changing at \$0 per widget. That is, profit is not changing at this point in time.