Math 229
Exam 3 - Solutions **Name:**______________________________
03/09/2007

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. I will give credit to each problem proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Be sure to simplify answers when possible. Also, make sure to follow directions carefully on each problem.

1. (7 points each)

   (a) Find the third derivative of $f(x)$, given that $f(x) = \dfrac{3}{x^2}$

   Solution:
   First, notice that $f(x) = \dfrac{3}{x^2} = 3x^{-2}$
   Therefore, $f'(x) = -6x^{-3}$
   $f''(x) = 18x^{-4}$
   and $f'''(x) = -72x^{-5}$

   (b) Find the second derivative of $g(x)$, given that $g(x) = \dfrac{2x+1}{3x+2}$

   Solution:
   Using the quotient rule:
   $g'(x) = \dfrac{(2)(3x+2) - (2x+1)(3)}{(3x+2)^2} = \dfrac{6x+4-6x-3}{(3x+2)^2} = \dfrac{1}{(3x+2)^2} = (3x+2)^{-2}$
   Therefore, using the chain rule, $g''(x) = -2(3x+2)^{-3}(3) = \dfrac{-6}{(3x+2)^3}$

   (c) Find the second derivative of $h(x)$, given that $h(x) = (1-x^2)^7$

   Solution:
   Using the chain rule: $h'(x) = 7(1-x^2)^6(-2x) = -14x(1-x^2)^6$.
   Then, applying the product and chain rule: $h''(x) = (-14)(1-x^2)^6 + (-14x)(6)(1-x^2)^5(-2x)$
   $= -14(1-x^2)^6 + 168x^2(1-x^2)^5$

2. (5 points each) Determine whether the following statements are True or False. Write a brief explanation to justify your answer.

   (a) If $f'(a) = 0$ and $f''(a) < 0$, then $(a, f(a))$ is a relative minimum of the function $f(x)$.

   Solution: False

   Since $f'(a) = 0$, $(a, f(a))$ is a critical point of $f(x)$. Also, since $f''(a) < 0$, $f(x)$ is concave down when $x = a$. Therefore, using the second derivative test, $(a, f(a))$ is actually a relative maximum of $f(x)$.

   (b) If $f'(x) > 0$ for $a \le x \le b$, then $(a, f(a))$ is an absolute minimum for $f(x)$ on $[a, b]$.

   Solution: True

   Since $f'(a) > 0$ for $a \le x \le b$, $f(x)$ is increasing on the interval $[a, b]$. Therefore, the smallest value of $f(x)$ on the interval $[a, b]$ must occur at the left endpoint. That is, $(a, f(a))$ is an absolute minimum for $f(x)$ on $[a, b]$.

3. (6 points each) Let $f(x) = x^4 - 8x^3 + 16x^2$

   (a) Find the x and y intercepts of $f(x)$.

   To find the y-intercept, we evaluate $f(x)$ when $x = 0$. Since $f(0) = 0$, $(0, 0)$ is the y-intercept.

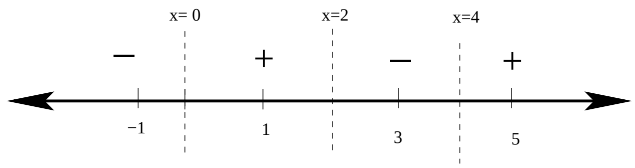
   To find the x-coordinates of the x-intercepts, we solve the equation $x^4 - 8x^3 + 16x^2 = 0$. Factoring this, we have $x^2(x^2 - 8x + 16) = 0$, or $x^2(x - 4)(x - 4) = 0$, so $x = 0$ and $x = 4$ are the solutions to this equation. Thus, $(0, 0)$ and $(4, 0)$ are the x-intercepts of $f(x)$.

   (b) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

   First notice that $f'(x) = 4x^3 - 24x^2 + 32x = 4x(x^2 - 6x + 8) = 4x(x - 4)(x - 2)$.

   Therefore, the critical points occur when $x = 0$, $x = 2$, and $x = 4$.

   Using sign analysis:



   Thus $f(x)$ is increasing on: $(0, 2) \cup (4, \infty)$

   And $f(x)$ is decreasing on: $(-\infty, 0) \cup (2, 4)$

   (c) Find and classify the relative extrema of $f(x)$.

   First notice that $f(0) = 0$, $f(2) = 2^4 - 8(2)^3 + 16(2)^2 = 16 - 64 + 64 = 16$, and $f(4) = 4^4 - 8(4)^3 + 16(4)^2 = 256 - 512 + 256 = 0$.

   From our sign analysis above: $(0, 0)$ and $(4, 0)$ are relative minima, while $(2, 16)$ is a relative maximum.

   (d) Find the equation of the tangent line to $f(x)$ when $x = 1$.

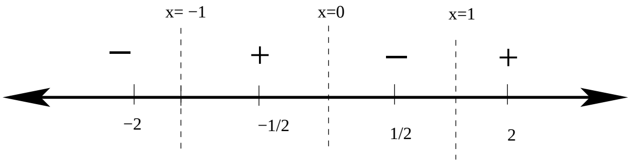
   To find the tangent line, we need a point and a slope. To find the point, we evaluate $f(1) = 1 - 8 + 16 = 9$, giving us the point $P = (1, 9)$. To find the slope, we evaluate $f'(1) = 4 - 24 + 32 = 12$, giving us the slope $m = 12$. Therefore, using the point slope formula, $y - 9 = 12(x - 1)$, or $y = 12x - 3$.

4. Given that $f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3$ and $f'(x) = x^4 - 2x^2$:

(a) (8 points) Find the intervals where $f(x)$ is concave up and the intervals where $f(x)$ is concave down.

First, notice that $f''(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$. Therefore, the key values of $f''(x)$ occur when $4x(x+1)(x-1) = 0$, or at $x = 0$, $x = -1$, and $x = 1$.

Using sign analysis:



Thus $f(x)$ is concave up on: $(-1, 0) \cup (1, \infty)$

And $f(x)$ is concave down on: $(-\infty, -1) \cup (0, 1)$

(b) (6 points) Find the coordinates of the inflection points of $f(x)$.

Notice that $f(0) = 0$, $f(1) = \frac{1}{5} - \frac{2}{3} = \frac{3}{15} - \frac{10}{15} = -\frac{7}{15}$, and $f(-1) = -\frac{1}{5} + \frac{2}{3} = -\frac{3}{15} + \frac{10}{15} = \frac{7}{15}$

Therefore, the inflection points are: $(0, 0)$, $(1, -\frac{7}{15})$, and $(-1, \frac{7}{15})$.

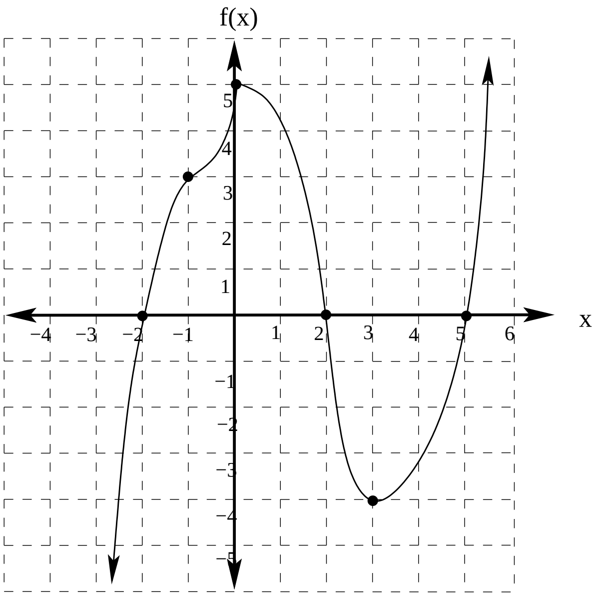
5. (15 points) Carefully draw the graph of a function satisfying the following conditions:

x-intercepts: $(-2, 0), (2, 0), (5, 0)$; y-intercept: $(0, 5)$

Increasing on $(-\infty, 0) \cup (3, \infty)$ and Decreasing on $(0, 3)$

Concave Up on $(-1, 0) \cup (2, \infty)$ and Concave Down on $(-\infty, -1) \cup (0, 2)$

$f(-1) = 3$, and $f(3) = -4$.



6. Suppose the daily cost for producing x widgets is given by $C(x) = 5x^2 - 20x + 500$, where $C(x)$ is in dollars, and a maximum of 20 widgets can be produced each day.

(a) (8 points) Find the production level which minimizes the daily costs. Also find the daily cost at this production level.

We are looking for the absolute minimum of $C(x)$ on the interval $[0, 20]$. To find this, we first compute $C'(x) = 10x - 20$, and notice that it has a single critical point when $10x - 20 = 0$, or when $x = 2$.

Since the absolute extrema of a function on an interval must occur at either a critical point or one of the endpoints, we check to see what the cost is at each of these points of interest: $C(0) = 500$, $C(2) = 5(4) - 20(2) + 500 = 20 - 40 + 500 = 480$, and $C(20) = 5(400) - 20(20) + 500 = 2000 - 400 + 500 = 2100$.

Therefore, the cost is minimized when 2 widgets are produced at a cost of \$480.

(b) (8 points) Find the production level which minimizes the **average** cost per widget. Also find the average cost per sidget at this production level.

Here, we are interested in minimizing the average cost per widget produced rather than the total cost of production. To do this, we want to find the absolute minimum of the average cost function $\overline{C}(x) = \frac{C(x)}{x} = \frac{5x^2 - 20x + 500}{x} = 5x - 20 + 500x^{-1}$ on the interval $[0, 20]$.

We use the same procedure as above:
$\overline{C}'(x) = 5 - 500x^{-2} = 5 - \frac{500}{x^2} = \frac{5x^2 - 500}{x^2}$, which has critical points of both types. $x = 0$ makes the derivative undefined, and if $5x^2 - 500 = 0$, then $5x^2 = 500$, so $x^2 = 100$, or $x = \pm 10$. Notice that $x = -10$ is not in our interval of interest.

Since the absolute extrema of a function on an interval must occur at either a critical point or one of the endpoints, we check to see what the average cost is at each of these points of interest:
$\overline{C}(0)$ is undefined (division by zero).
$\overline{C}(10) = 5(10) - 20 + \frac{500}{10} = 50 - 20 + 50 = 80$
$\overline{C}(20) = 5(20) - 20 + \frac{500}{20} = 100 - 20 + 25 = 105$
Therefore, the absolute minimum average cost of \$80 per widget occurs when 10 widgets are produced.