Math 229
Exam 4 - Solutions **Name:**______________________________
04/18/2007

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. I will give credit to each problem proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Be sure to simplify answers when possible. Also, make sure to follow directions carefully on each problem.

1. (2 points each) Determine whether the following are True or False:

   (a) $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) + \ln(x-1)$

   False. Notice that $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) - \ln(x-1)$.

   (b) $e^{\ln\left(x^2+1\right)} = x^2+1$

   True. This is the inverse function property of exponential and log functions.

   (c) $e^{x^2}\cdot e^{3x} = e^{3x^3}$

   False. In fact, $e^{x^2}\cdot e^{3x} = e^{x^2+3x}$. The exponents **add** rather than multiply here.

   (d) $\dfrac{\ln(4x)}{\ln(2x)} = \ln 2$

   False. This is not a legal simplification. For example, if $x = 1$, then $\dfrac{\ln(4x)}{\ln(2x)} = \dfrac{\ln(4)}{\ln(2)} = 2$, while $\ln 2 \approx .6931$.

   (e) $\ln\left(e^{x^2} - 4\right) = x^2 - \ln 4$

   False. The inverse function property does not apply here because of the subtracton operation. In fact, if $x = 2$, $\ln\left(e^{2^2} - 4\right) \approx 3.924$, while $2^2 - \ln 4 \approx 2.614$

2. Suppose that a culture of bacteria that initially has 500 cells grows to 10,000 cells in 12 hours.

   (a) (5 points) Find a function $f(t)$ that gives the number of cells in the culture as a function of time (in hours), assuming that this population grows continuously and exponentially.

   Using the continuous exponential growth equation, $A = Pe^{rt}$, we see that $A = 10,000$, $p = 500$, and $t = 12$, or $10,000 = 500e^{12r}$. Therefore, $20 = e^{12r}$, so $\ln(20) = 12r$, and hence $r \approx .2496$. Therefore, our function modeling the growth of this bacterial culture is: $f(t) = 500e^{.2496t}$.

   (b) (5 points) How long will it take for the culture to reach 1,000,000 cells?

   Using the function $f(t) = 500e^{.2496t}$ found above, we solve $1,000,000 = 500e^{.2496t}$ for t. Then $2000 = e^{.2496t}$, so $\ln(2000) = .2496t$, so $t = \frac{\ln(2000)}{.2496} \approx 30.45$ hours

3. (7 points) Find the interest rate needed for an investment of \$1,000 to double in 7 years if the interest is compounded monthly.

   We will use the compound interest equation, $A = P(1+\frac{r}{n})^{nt}$. Here, we see that $P = \$1000$, $A = \$2000$, $t = 7$, and $n = 12$. To find the rate, we must solve the equation:

   $2000 = 1000(1+\frac{r}{12})^{(7)(12)}$, or $2 = (1+\frac{r}{12})^{84}$. Taking the natural logarithm of both sides, we obtain $\ln 2 = \ln\left[(1+\frac{r}{12})^{84}\right] = 84\ln(1+\frac{r}{12})$, therefore, $\frac{\ln 2}{84} = \ln(1+\frac{r}{12})$.

   If we exponentiate both sides of this equation, we get $e^{\frac{\ln 2}{84}} = 1+\frac{r}{12}$. Therefore, $(e^{\frac{\ln 2}{84}}-1)12 = r$. Hence $r \approx .09943$, or a rate of about 9.9% is needed for our investment to double in 7 years.

4. (6 points each) Compute the derivatives of the following functions. You do not need to simplify your answers.

   (a) $f(x) = \ln\left(\dfrac{x^2}{(2x-1)^3}\right)$
   Rather than just jumping in and differentiating, we can make our job a lot easier by simplifying $f(x)$ using properties of logarithms.
   $f(x) = \ln(x^2) - \ln(2x-1)^3 = 2\ln x - 3\ln(2x-1)$
   Then $f'(x) = 2\cdot\frac{1}{x} - 3\cdot\frac{2}{2x-1} = \frac{2}{x} - \frac{6}{2x-1}$

   (b) $g(x) = e^{e^{2x^3}}$
   Using the chain rule for exponential functions two times, we see:
   $g'(x) = e^{e^{2x^3}}\cdot e^{2x^3}\cdot 6x^2$

   (c) $h(x) = \ln(x^2+1)e^{x^3}$
   By the product rule: $h'(x) = \frac{2x}{x^2+1}e^{x^3} + \ln(x^2+1)(3x^2)e^{x^3}$.

   (d) $l(x) = (3x-2)^4(x^2-3x)^{\frac{5}{2}}(x^3-1)^{\frac{4}{3}}$
   Using logarithmic differentiation, we define $g(x) = \ln(l(x)) = \ln\left[(3x-2)^4(x^2-3x)^{\frac{5}{2}}(x^3-1)^{\frac{4}{3}}\right]$
   simplifying, $g(x) = 4\ln(3x-2) + \frac{5}{2}\ln(x^2-3x) + \frac{4}{3}\ln(x^3-1)$
   Then $g'(x) = 4\frac{3}{3x-2} + \frac{5}{2}\frac{2x-3}{x^2-3x} + \frac{4}{3}\frac{3x^2}{x^3-1}$
   $= \frac{12}{3x-2} + \frac{10x-15}{2x^2-6x} + \frac{4x^2}{x^3-1}$
   Therefore, $l'(x) = l(x)g'(x) = (3x-2)^4(x^2-3x)^{\frac{5}{2}}(x^3-1)^{\frac{4}{3}}\left[\dfrac{12}{3x-2} + \dfrac{10x-15}{2x^2-6x} + \dfrac{4x^2}{x^3-1}\right]$

5. (7 points) Find the tangent line to $f(x) = 2xe^{2x-4}$ when $x = 2$

   First, we find the y coordinate of the point of tangency by computing $f(2) = 2(2)e^{2(2)-4} = 4e^0 = 4$, so $P(2,4)$ is the point of tangency.

   Next, to find the slope of the tangent line, we compute $f'(x) = 2e^{2x-4} + (2x)(2)e^{2x-4}$, and evaluate when $x = 2$, $m = f'(2) = 2e^{2(2)-4} + (2)(2)(2)e^{2(2)-4} = 2e^0 + 8e^(0) = 2 + 8 = 10$.

   Finally, using point slope with $P(2,4)$ and $m = 10$,

   $y - 4 = 10(x-2)$, or $y = 10x - 16$ is the equation of the tangent line when $x = 2$.

6. (8 points) Determine the intervals where the function $g(t) = t^4e^{2t}$ in increasing and the intervals where it is decreasing.

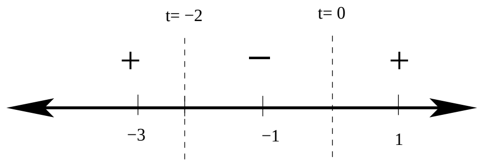
   To find the intervals where $g(t)$ is increasing and those where $g(t)$ are decreasing, we analyze the first derivative of $g(t)$:

   $g'(t) = 4t^3e^{2t} + t^4(2)e^{2t} = 4t^3e^{2t} + 2t^4e^{2t} = 2t^3e^{2t}(2+t)$.

   To find the critical points, we solve the equation: $2t^3e^{2t}(2+t) = 0$

   We see that if $2t^3 = 0$, then $t = 0$, $e^{2t} = 0$ has no solution, and $2 + t = 0$ when $t = -2$, so we have two critical points, $t = 0$ and $t = -2$.

   We use test points to determine the sign of the first derivative on each of the intervals between our critical points, as indicated in the following sign-testing diagram:



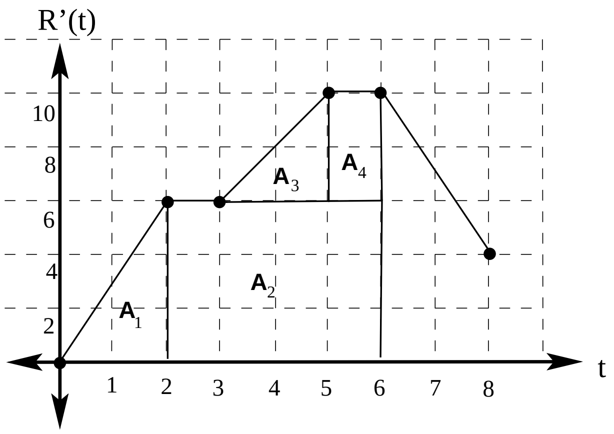
   Therefore, $g(t)$ is increasing on the intervals $(-\infty, -2) \cup (0, \infty)$, and $g(t)$ is decreasing on the interval $(-2, 0)$.

7. (4 points) Suppose $\int_0^2 f(x)\ dx = 4$ and $\int_0^2 g(x)\ dx = 2$. Find $\int_0^2 2f(x) - g(x)\ dx$

   Using the properties of the definite integral:

   $$\int_0^2 2f(x) - g(x)\ dx = \int_0^2 2f(x)\ dx - \int_0^2 g(x)\ dx = 2\int_0^2 f(x)\ dx - \int_0^2 g(x)\ dx = 2(4) - 2 = 6.$$

8. (8 points) Suppose marginal revenue, $R'(t)$ is given by the graph below, where t in in months and $R'(t)$ is in \$1000s of dollars per month. Find the total revenue from $t = 0$ to $t = 6$.



   To find the total revenue from $t = 0$ to $t = 6$, we need to find the area under the marginal revenue function $R'(t)$ between $t = 0$ adn $t = 6$. We do not have a formula for $R'(t)$, but we do not need one, since we can actually use geometry to find this area (see the figure above).

   $A_1 = 3$ squares, $A_2 = 12$ squares, $A_3 = 2$ squares, and $A_4 = 2$ squares.

   Since the total area under the curve from $t = 0$ to $t = 6$ is 19 squares, and each square corresponds to \$2000 of revenue, the total revenue for this time period is \$38,000.

9. (6 points each) Evaluate the following integrals:

(a) $\displaystyle\int 5x^4 - x^{\frac{3}{2}}\ dx = x^5 - \frac{2}{5}x^{\frac{5}{2}} + C$

(b) $\displaystyle\int \frac{4x^2+2}{x^3}\ dx$

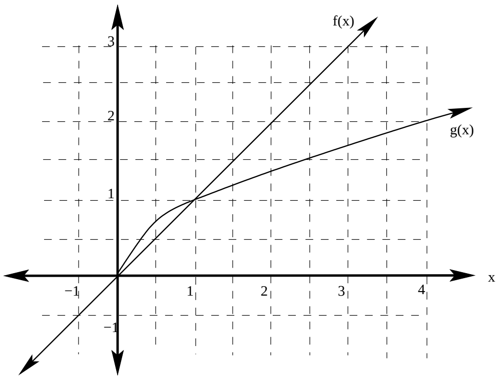
We first simplify this in order to obtain a form of the function that is easier to antdifferentiate and then carry out the integration:

$\displaystyle\int \frac{4x^2}{x^3} + \frac{2}{x^3}\ dx = \int \frac{4}{x} + 2x^{-3}\ dx = 4\ln|x| - x^{-2} + C$

(c) $\displaystyle\int_0^3 4x^2 - e^{3x}\ dx = \frac{4}{3}x^3 - \frac{1}{3}e^{3x}|_3^0 = \left(\frac{4}{3}(3)^3 - \frac{1}{3}e^{(3)(3)}\right) - \left(\frac{4}{3}(0)^3 - \frac{1}{3}e^{(0)(3)}\right)$

$= \left(36 - \frac{1}{3}e^9\right) - \left(0 - \frac{1}{3}\right) = 36 + \frac{1}{3} - \frac{1}{3}e^9 \approx -2664.67.$

10. (8 points) Find the area of the region enclosed by the graphs $f(x) = x$ and $g(x) = \sqrt{x}$.



Using algebra, we see that if $f(x) = g(x)$, then $x = \sqrt{x}$, or, squaring both sides, $x^2 = x$. That is, $x^2 - x = 0$, or $x(x-1) = 0$. Therefore, $x = 0$ or $x = 1$. That is, these two functions cross when $x = 0$ and when $x = 1$. From the graph above, we see that there is a single region enclosed by these two functions, and that $g(x)$ is greater than $f(x)$ when x is between 0 and 1.

Therefore, $\displaystyle A = \int_0^1 g(x) - f(x)\ dx = \int_0^1 x^{\frac{1}{2}} - x\ dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2|_0^1$

$= \left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{2}(1)^2\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{2}(0)^2\right) = \left(\frac{2}{3} - \frac{1}{2}\right) - (0 - 0) = \frac{1}{6}$