Solving a System of 3 Equations in 3 Unknowns Using Matrices

Example 1: Given the system of equations:

$$\begin{cases} 3x + 2y = 5 \\ -x + 4z = -2 \\ 2x + y = 10 \end{cases}$$

We will solve this system by changing to matrix form and transforming the matrix:

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ -1 & 0 & 4 & -2 \\ 2 & 1 & 0 & 10 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -5 \\ -1 & 0 & 4 & -2 \\ 2 & 1 & 0 & 10 \end{array}\right] \xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -5 \\ -1 & 0 & 4 & -2 \\ 0 & 1 & 8 & 6 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -5 \\ 0 & 1 & 4 & -7 \\ 0 & 1 & 8 & 6 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -5 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 4 & 13 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 4 & 13 \end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 4 & 13 \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 4 & 13 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & \frac{13}{4} \end{array}\right]$$

Therefore, the solution to this system is $x = 15$, $y = -20$, and $z = \frac{13}{4}$
Example 2: Given the system of equations:

$$\begin{cases} 2x + y - z = 2 \\ 3x + 2z = 4 \\ x - y + 3z = 2 \end{cases}$$

We will solve this system by changing to matrix form and transforming the matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 3 & 0 & 2 & 4 \\ 1 & -1 & 3 & 2 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 3 & 0 & 2 & 4 \\ 2 & 1 & -1 & 2 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 3 & -7 & -2 \\ 2 & 1 & -1 & 2 \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 3 & -7 & -2 \\ 0 & 3 & -7 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 3 & -7 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 1 & -\frac{7}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{7}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that there is nothing else we can do to reduce this matrix. Since we have a row of zeroes, this system of equations does not have a unique solution. Since $0 = 0$ is always true, we know that we have infinitely many solutions - actually, a line of solutions. But we can do a bit better than this. Interpreting the reduced form of the matrix found above, we have the equations:
$x + \frac{2}{3}z = \frac{4}{3}$, or $x = \frac{4}{3} - \frac{2}{3}z$
and $y - \frac{7}{3}z = -\frac{2}{3}$, or $y = -\frac{2}{3} + \frac{7}{3}z$
Letting $z = t$ gives us the following equations:

$$\begin{cases} x = \frac{4}{3} - \frac{2}{3}t \\ y = -\frac{2}{3} + \frac{7}{3}t \\ z = t \end{cases}$$

That is, all solutions to this system of equations have the form: $(\frac{4}{3} - \frac{2}{3}t, -\frac{2}{3} + \frac{7}{3}t, t)$.
For example, when $t = 0$, we have the solution $(\frac{4}{3}, -\frac{2}{3}, 0)$.