Math 229
Practice Exam 3

1. Fine the first, second, and third derivative of the following functions:

   (a) $f(x) = 3x^4 - 4x^2 - 10x + 17$
   $f'(x) = 12x^3 - 8x - 10$
   $f''(x) = 36x^2 - 8$
   $f'''(x) = 72x$

   (b) $g(x) = \sqrt{3x-4}(= (3x-4)^{\frac{1}{2}})$
   $g'(x) = \frac{1}{2}(3x-4)^{-\frac{1}{2}}(3) = \frac{3}{2}(3x-4)^{-\frac{1}{2}}$
   $g''(x) = \left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)(3x-4)^{-\frac{3}{2}}(3) = -\frac{9}{4}(3x-4)^{-\frac{3}{2}}$
   $g'''(x) = \left(-\frac{3}{2}\right)\left(-\frac{9}{4}\right)(3x-4)^{-\frac{5}{2}}(3) = \frac{81}{8}(3x-4)^{-\frac{5}{2}}$

   (c) $h(x) = (x^2+1)^{10}$
   $h'(x) = 10(x^2+1)^9(2x) = (20x)(x^2+1)^9$
   $h''(x) = (20)(x^2+1)^9 + (20x)(9)(x^2+1)^8(2x) = 20(x^2+1)^9 + 360x^2(x^2+1)^8$
   $h'''(x) = 20(9)(x^2+1)^8(2x) + 720x(x^2+1)^8 + 360x^2(8)(x^2+1)^7(2x)$
   $= 360x(x^2+1)^8 + 720x(x^2+1)^8 + 5760x^3(x^2+1)^7 = 1080x(x^2+1)^8 + 5760x^3(x^2+1)^7$

2. True or False:

   (a) If $f'(a) = 0$, then $(a, f(a))$ is either a relative maximum or a relative minumum of $f(x)$.
   False - for example, if $f(x) = x^3$, $f'(x) = 3x^2$ so $x = 0$ is a critical point, but it is actually an inflection point, but not a relative maximum or a relative minimum.

   (b) If $f'(a) = 0$, and $f''(a) > 0$, then $(a, f(a))$ is a relative minumum of $f(x)$.
   True - The is one of the conclusions of the Second Derivative Test.

   (c) The absolute maximum of a function $f(x)$ on an interval $[a, b]$ must occur when $x = a$, when $x = b$, or at a critical point of f inside the interval $[a, b]$.
   True - see page 783 in your text for a complete explaination.

3. Let $f(x) = \frac{1}{4}x^4 - x^3$

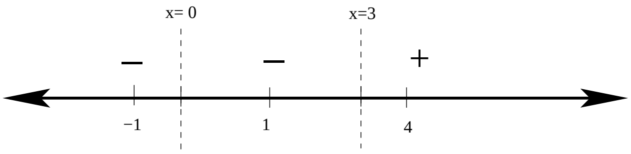
(a) Find the x and y intercepts of $f(x)$.

y-intercept: When $x = 0$, $f(x) = \frac{1}{4}0^4 - 0^3 = 0$, so $(0, 0)$ is the y-intercept.

x-intercepts: If $f(x) = \frac{1}{4}x^4 - x^3 = 0$, then $x^3(\frac{1}{4}x - 1) = 0$, so either $x^3 = 0$, so $x = 0$, or $\frac{1}{4}x - 1 = 0$, so $\frac{1}{4}x = 1$, thus $x = 4$, so $(0, 0)$ and $(4, 0)$ are the x-intercepts.

(b) Find the intervals where $f(x)$ is increasing and those where $f(x)$ is decreasing.

Since $f'(x) = x^3 - 3x^2$, the critical points occur when $x^3 - 3x^2 = 0$, or when $x^2(x - 3) = 0$, therefore, the critical values are $x = 0$ and $x = 3$. We do sign analysis using the test points $x = -1$, $x = 1$, and $x = 4$ and observe that $f'(-1) = -1 - 3 = -4$, $f'(1) = 1 - 3 = -2$, and $f'(4) = 64 - 48 = 16$.



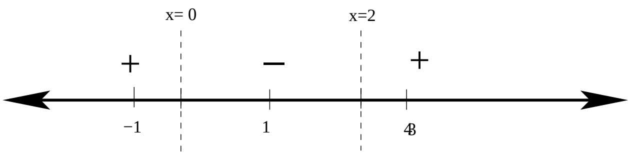
Therefore, $f(x)$ is increasing on $(3, \infty)$ and decreasing on $(-\infty, 0) \cup (0, 3)$.

(c) Find and classify the relative extrema of $f(x)$.

Since the first derivative does not change signs when we cross $x = 0$, there is not an extremum there. On the other hand, the derivative goes from negative to positive when we move across $x = 3$, so the point $(3, f(3)) = (3, -\frac{27}{4})$ is a relative minimum.

(d) Find the intervals where $f(x)$ is concave up and those where $f(x)$ is concave down.

Since $f''(x) = 3x^2 - 6x$, the critical points occur when $3x^2 - 6x = 0$, or when $3x(x - 2) = 0$, therefore, the critical values are $x = 0$ and $x = 2$. We do sign analysis using the test points $x = -1$, $x = 1$, and $x = 3$ and observe that $f''(-1) = 3 + 6 = 9$, $f''(1) = 3 - 6 = -3$, and $f''(3) = 27 - 18 = 9$.

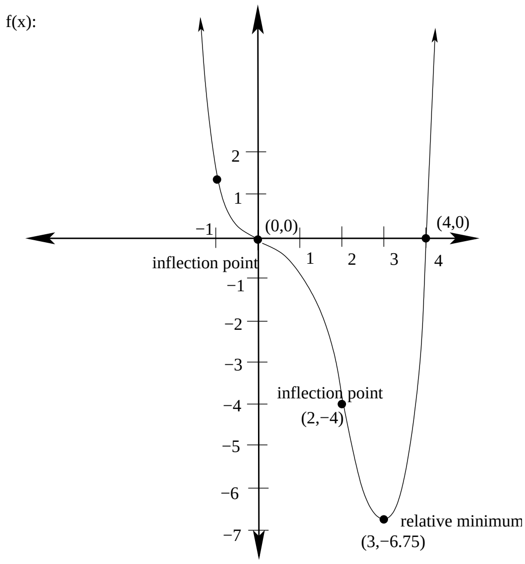


Therefore, $f(x)$ is concave down on $(0, 2)$ and concave up on $(-\infty, 0) \cup (2, \infty)$.

(e) Find any inflection points of $f(x)$.

Notice that the second changes signs both when $x = 0$ is crossed and when $x = 2$ is crossed. Therefore, both $(0, 0)$, and $(2, f(2)) = (2, -4)$ are inflection points.

(f) Graph $f(x)$. Be sure to label all relative extrema, intercepts, and inflection points.



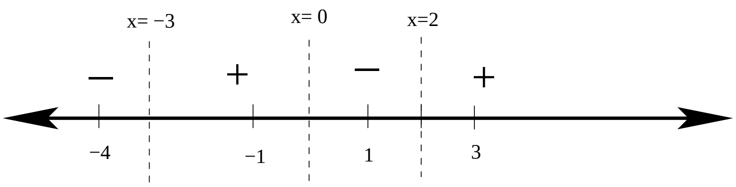
4. Let $f(x) = \frac{3}{4}x^4 + x^3 - 9x^2 + 12$

(a) Find the intervals where $f(x)$ is increasing and those where $f(x)$ is decreasing.

First we compute the first derivative: $f'(x) = 3x^3 + 3x^2 - 18x$

Next, we find the critical points. Notice that there are no points at which the first derivative is undefined, so all critical points can be found by solving the equation: $3x^3 + 3x^2 - 18x = 0$. Factoring, we have $3x(x^2 + x - 6) = 0$, which further factors to give $3x(x+3)(x-2) = 0$, which has solutions $x = 0, x = -3$, and $x = 2$.

We now test the sign of the first derivative on the regions between the critical points:



Therefore, $f(x)$ is decreasing on $(-\infty, -3) \cup (0, 2)$, and increasing on $(-3, 0) \cup (2, \infty)$.

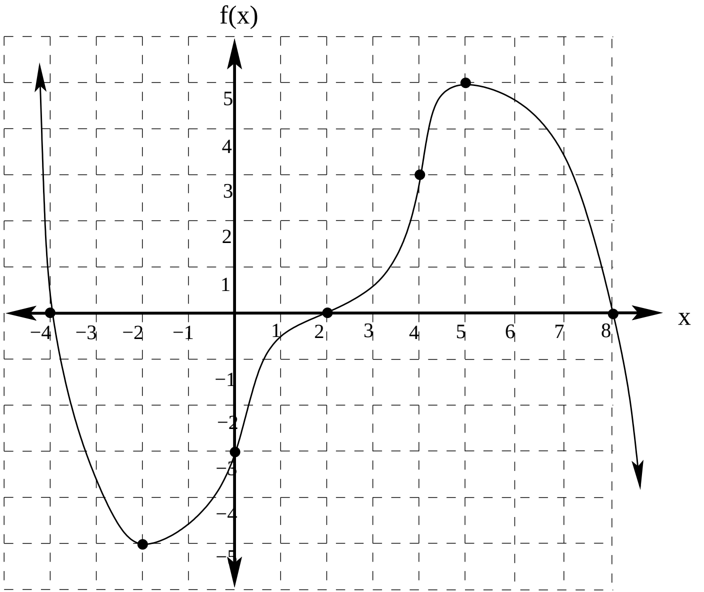
(b) Find and classify the relative extrema of $f(x)$.

Again looking at our sign analysis table, we see that the function has relative minima when $x = -3$ and $x = 2$, and a relative maximum when $x = 0$. That is, $(-3, -35.25)$ and $(2, -4)$ are relative minima, and $(0, 12)$ is a relative maximum.

(c) Find the equation for the tangent line to $f(x)$ when $x = 2$.

Notice that $f(2) = -4$, and $f'(2) = 0$, so we have a horizontal line through the point $(2, -4)$, which has equation $y = -4$.

5. Sketch a function in the space provided that satisfies the following conditions:
Domain: $(-\infty, \infty)$, x-intercepts: $(-4, 0), (2, 0), (8, 0)$, y-intercept: $(0, -3)$
Increasing on: $(-2, 5)$, Decreasing on: $(-\infty, -2) \cup (5, \infty)$
Concave up on: $(-\infty, 0) \cup (2, 4)$, Concave down on: $(0, 2) \cup (4, \infty)$
Local Max: $(5, 5)$, Local Min: $(-2, -5)$, Inflection Points: $(0, -3), (2, 0), (4, 3)$



6. Suppose the daily cost and revenue for producing x widgets are given by the functions:
$C(x) = 750 - 3x + .005x^2$ and $R(x) = 825 + 2x - .005x^2$ for $0 \le x \le 400$.

(a) Find the production level which minimizes the daily production costs. Also find the cost at this production level.
Notice that $C'(x) = -3 + .01x$, which is zero when $.01x = 3$, or when $x = 300$. Since we are restricted to values of x between 0 and 400, we check the following cost levels:
$C(0) = 750 - 3(0) + .005(0)^2 = 750$
$C(300) = 750 - 3(300) + .005(300)^2 = 750 - 900 + .005(90000) = 300$
$C(400) = 750 - 3(400) + .005(400)^2 = 750 - 1200 + .005(160000) = 350$
Thus daily production cost is minimized at \$300 when 300 widgets are produced.

(b) Find the production level which maximizes daily **profits**. Also find the amount of profit at this production level.
Recall that $P(x) = R(x) - C(x) = (825 + 2x - .005x^2) - (750 - 3x + .005x^2) = 75 + 5x - .01x^2$
Therefore, $P'(x) = 5 - .02x$, which is zero when $.02x = 5$, or when $x = 250$.
We check the profit level at both the endpoints and the critical point in order to find the absolute maximum profit.
$P(0) = 75 + 5(0) - .01(0)^2 = 75$
$P(250) = 75 + 5(250) - .01(250)^2 = 75 + 1250 - 625 = 700$
$P(400) = 75 + 5(400) - .01(400)^2 = 75 + 2000 - 1600 = 475$
Thus profit is maximized at \$700 when 250 widgets are produced.