Math 229, Section 13
Practice Exam 4 - Solutions **Name:**______________________________

1. Find the exact value of the following logarithmic expressions:

   (a) $\log_2(32) = 5$ [since $2^5 = 32$] (c) $\log_5(1) = 0$ [since $5^0 = 1$]

   (b) $\log_3\left(\frac{1}{27}\right) = -3$ [since $3^{-3} = \frac{1}{27}$] (d) $\log_4 32 = 2.5$ [since $2^5 = \left(4^{\frac{1}{2}}\right)^5 = 4^{\frac{5}{2}}$]

2. Use the laws of logarithms to simplify the expression: $\ln\left(\frac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right)$

$$\ln\left(\frac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right) = \ln x^2 + \ln(x-1)^{\frac{5}{2}} - \ln(x-4)^3 = 2\ln x + \frac{5}{2}\ln(x-1) - 3\ln(x-4)$$

3. (a) Suppose you invest $10,000 in a savings account that pays 3% annual interest compounded monthly. How much money will be in the account after 6 years?
   $A = P(1+\frac{r}{n})^{nt} = 10,000(1+\frac{.03}{12})^{(12)(6)} = 10,000(1.0025)^{72} \approx \$11,969.48$

   (b) How long would it take $5,000 invested at 6% annual interest compounded continuously to triple?
   The continuous interest formula is: $A = Pe^{rt}$. So we have $15,000 = 5,000e^{.06t}$, or $3 = e^{.06t}$, which makes sense since we want our initial investment to triple.
   Taking the natural logarithm of both sides gives: $\ln 3 = \ln(e^{.06t}) = .06t$, so $t = \frac{\ln 3}{.06} \approx 18.31$ years.

   (c) Find the interest rate needed for an investment of $2,000 to double in 6 years if the interest is comounded quarterly.
   Using the compound interest formula $A = P(1+\frac{r}{n})^{nt}$, we have $4,000 = 2,000(1+\frac{r}{4})^{(4)(6)}$, or $2 = (1+\frac{r}{4})^{24}$. Taking the natural log of both sides, $\ln 2 = \ln(1+\frac{r}{4})^{24} = 24\ln(1+\frac{r}{4})$, so $\frac{\ln 2}{24} = \ln(1+\frac{r}{4})$. Exponentiating both sides, we than have $e^{\frac{\ln 2}{24}} = e^{\ln(1+\frac{r}{4})} = 1+\frac{r}{4}$.
   Hence $e^{\frac{\ln 2}{24}} - 1 = \frac{r}{4}$, therefore $4(e^{\frac{\ln 2}{24}} - 1) = r$, so $r \approx .1172$, or %11.72

4. Compute the derivatives of the following functions. You do not need to simplify your answers.

   (a) $f(x) = e^{3x^2}$
   $f'(x) = e^{3x^2} \cdot 6x = 6xe^{3x^2}$

   (b) $g(x) = \ln(3x^2 - 4x + 6)$
   $g'(x) = \frac{1}{3x^2-4x+6} \cdot (6x-4) = \frac{6x-4}{3x^2-4x+6}$

   (c) $h(x) = (x^2+1)e^{x^2+1}$
   By the product rule, $h'(x) = 2xe^{x^2+1} + (x^2+1)e^{x^2+1} \cdot (2x) = 2xe^{x^2+1}(1+x^2+1) = 2xe^{x^2+1}(x^2+2)$

   (d) $k(x) = x^2\ln(e^x+1)$
   Again using the product rule, $k'(x) = 2x\ln(e^x+1) + x^2\frac{1}{e^x+1} \cdot e^x = 2x\ln(e^x+1) + \frac{x^2e^x}{e^x+1}$

(e) $l(x) = (3x^2+1)^5(x^2-1)^{\frac{3}{2}}(4x+3)^{\frac{5}{3}}$

Using logarithmic differentiation, we define:

$g(x) = \ln\left((3x^2+1)^5(x^2-1)^{\frac{3}{2}}(4x+3)^{\frac{5}{3}}\right) = 5\ln(3x^2+1) + \frac{3}{2}\ln(x^2-1) + \frac{5}{3}\ln(4x+3)$

Differentiating this, we get:

$g'(x) = (5)\frac{1}{3x^2+1}(6x) + \left(\frac{3}{2}\right)\frac{1}{x^2-1}(2x) + \left(\frac{5}{3}\right)\frac{1}{4x+3}(4) = \frac{30x}{3x^2+1} + \frac{3x}{x^2-1} + \frac{20}{12x+9}$

Finally, recall that $l'(x) = g'(x)l(x) = \left(\frac{30x}{3x^2+1} + \frac{3x}{x^2-1} + \frac{20}{12x+9}\right)\left((3x^2+1)^5(x^2-1)^{\frac{3}{2}}(4x+3)^{\frac{5}{3}}\right)$

5. Find the tangent line to $f(x) = x\ln(2x)$ when $x = \frac{1}{2}$

First notice that $f(\frac{1}{2}) = \frac{1}{2}\ln(2\frac{1}{2}) = \frac{1}{2}\ln(1) = \left(\frac{1}{2}\right)(0) = 0$, so a point on our line is $P = (\frac{1}{2}, 0)$.

Next, to find the slope of the tangent line, we differentiate using the product rule:

$f'(x) = (1)\ln(2x) + x\frac{1}{2x}\cdot 2 = \ln(2x) + \frac{2x}{2x} = \ln(2x) + 1.$

Then $f'(\frac{1}{2}) = \ln(2\cdot\frac{1}{2}) + 1 = \ln(1) + 1 = 0 + 1 = 1$, so $m = 1$.

Finally, using the point slope equation, $y - y_1 = m(x - x_1)$, we have $y - 0 = 1(x - \frac{1}{2})$, or $y = x - \frac{1}{2}$

6. Find the absolute extrema of $g(t) = t^2e^{2t}$ on the interval $[-2, 2]$.

Recall that absolute extrema occur either at critical points of our function, or at the endpoints of our interval.

We first find the critical points of g:

$g'(t) = 2te^{2t} + t^2e^{2t}(2) = 2te^{2t} + 2t^2e^{2t} = (2t)(e^{2t})(1+t) = 0$

Since $g'(t)$ is continuous, the only critical points occur when the derivative is zero, or when $2t = 0$, $e^{2t} = 0$, or when $1 + t = 0$. Since e^{2t} is never zero, our solutions are $t = 0$, and $t = -1$.

Now, we can actually skip analyzing these critical points since classifying them is not necessary. So we just evaluate the original function on each critical point and on the two endpoints of the given interval and see what the highest and lowest values are:

$g(-2) = 4e^{-4} \approx .07326$

$g(2) = 4e^4 \approx 218.393$ – absolute maximum

$g(0) = 0e^0 = 0$ – absolute minimum

$g(-1) = 1e^{-2} \approx .1353$

7. Evaluate the following integrals:

(a) $\int 6x^3 - 4x^{\frac{1}{2}}\, dx = 6\left(\frac{1}{4}\right)x^4 - 4\left(\frac{2}{3}\right)x^{\frac{3}{2}} + C = \frac{3}{2}x^2 - \frac{8}{3}x^{\frac{3}{2}} + C$

(b) $\int \frac{4x^3 - 3x^2 + 2x}{2x^2}\, dx = \int \frac{4x^3}{2x^2} - \frac{3x^2}{2x^2} + \frac{2x}{2x^2}\, dx = \int 2x - \frac{3}{2} + \frac{1}{x}\, dx = x^2 - \frac{3}{2}x \ln|x| + C$

(c) $\int_{-1}^{1} 3x^5 - 4x^3\, dx = 3\left(\frac{1}{6}\right)x^6 - 4\left(\frac{1}{4}\right)x^4\Big|_{-1}^{1} = \frac{1}{2}x^6 - x^4\Big|_{-1}^{1} = \left(\frac{1}{2} - 1\right) - \left(\frac{1}{2} - 1\right)$

$= (-\frac{1}{2}) - (-\frac{1}{2}) = 0$

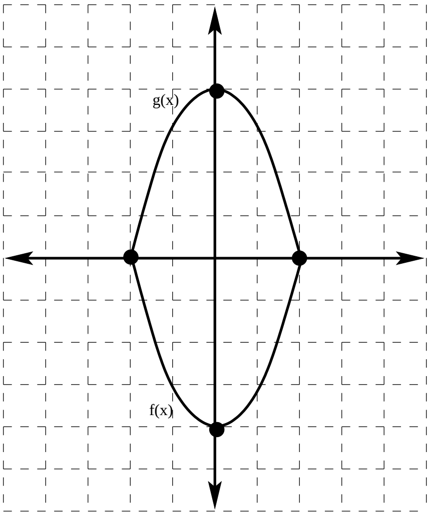
(d) $\int_0^4 e^{3x} + x^{-\frac{1}{2}}\, dx = \frac{1}{3}e^{3x} + 2x^{\frac{1}{2}}\Big|_0^4 = \left(\frac{1}{3}e^{12} + 2\sqrt{4}\right) - \left(\frac{1}{3}e^0 + 2\sqrt{0}\right)$

$= \frac{1}{3}e^{12} + 4 - \frac{1}{3} = \frac{1}{3}e^{12} + \frac{11}{3} \approx 54,255.263$

8. Find the average value of $f(x) = x^2 - \frac{1}{x^2}$ for $1 \leq x \leq 3$.

Recall that Average Value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \dfrac{1}{3-1}\displaystyle\int_1^3 x^2 - x^{-2}\,dx = \dfrac{1}{2}\left[\dfrac{1}{3}x^3 + x^{-1}\Big|_1^3\right] =$
$\dfrac{1}{2}\left[\left(\dfrac{27}{3} + \dfrac{1}{3}\right) - \left(\dfrac{1}{3} + 1\right)\right]$
$= \frac{1}{2}\left[\frac{28}{3} - \frac{4}{3}\right] = 4$

9. Find the area of the region enclosed by the graphs $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$.



To find the region between the graphs, we need to know where the functions meet.
Notice if $f(x) = g(x)$, then $x^2 - 4 = 4 - x^2$, or $2x^2 = 8$
Thus $x^2 = 4$, or $x = \pm 2$. Since $f(2) = g(2) = 4 - 4 = 0$, and $f(-2) = g(-2) = 4 - 4 = 0$,
$f(x)$ and $g(x)$ both contain the points $(-2, 0)$ and $(2, 0)$.
Also notice that $g(x)$ is greater than $f(x)$ for $-2 < x < 2$.

Then the area enclosed by $f(x)$ and $g(x)$ is given by:

$$A = \int_{-2}^{2} g(x) - f(x)\,dx = \int_{-2}^{2} (4 - x^2) - (x^2 - 4)\,dx = \int_{-2}^{2} 8 - 2x^2\,dx = 8x - \frac{2}{3}x^3\Big|_{-2}^{2}$$

$= \left[16 - \frac{2}{3}(8)\right] - \left[-16 + \frac{2}{3}(8)\right] = 32 - \frac{32}{3} = \frac{64}{3} \approx 21.333$