Math 229

Final Exam Practice Problems

1. Given that $f(x) = \sqrt{3-2x}$ and $g(x) = \dfrac{4}{3x-6}$

   (a) Find $f(-3)$ and $g(3a+2)$
   $f(-3) = \sqrt{3-2(-3)} = \sqrt{3+6} = \sqrt{9} = 3$
   $g(3a+2) = \frac{4}{(3)(3a+2)-6} = \frac{4}{9a+6-6} = \frac{4}{9a}$

   (b) Find $\dfrac{g}{f}(1)$
   $$\frac{g}{f}(1) = \frac{\frac{4}{3-6}}{\sqrt{3-2}} = \frac{\frac{4}{-3}}{1} = -\frac{4}{3}$$

   (c) Find $f \circ g(1)$
   $f \circ g(1) = f(g(1)) = f\left(\frac{4}{3-6}\right) = f\left(-\frac{4}{3}\right) = \sqrt{3-2\left(-\frac{4}{3}\right)} = \sqrt{3+\frac{8}{3}} = \sqrt{\frac{9}{3}+\frac{8}{3}} = \sqrt{\frac{17}{3}} = \frac{\sqrt{51}}{3}$

   (d) Find the domain of $fg(x)$? Give your answer in interval notation.
   To be in the domain of $fg(x)$, x must be in *both* the domain of f and the domain of g.
   The domain of f is all x such that $3-2x \geq 0$ (since we can't take the square root of a negative number).
   That is, $3 \geq 2x$, or $\frac{3}{2} \geq x$.
   The domain of g is all x such that $3x - 6 \neq 0$ (since we can't divide by zero. That is, we can't have $3x = 6$, or $x = 2$ is not allowed. Combining these, we see that the domain of $fg(x)$ is $(-\infty, \frac{3}{2}]$

2. The S. Claus Toy Company is working on a shipment of sleds that needs to ship at the end of the week. Since his workers are very inexpensive and his material costs are low, his overhead costs are only \$50 a day, and it only costs \$25 in parts and labor to make each sled. Based on a complete analysis of who has been naughty and who has been nice, Mr. Claus has determined that the daily demand equation for sleds is given by the equation $2p + 4x = 100$, where p is the price per sled in dollars, and x is the number of sleds sold each day.

   (a) Find an equation $C(x)$ the daily cost function for sled production, where x is the number of sleds made each day in the workshop.
   $C(x) = 25x + 50$

   (b) Find the daily revenue function, $R(x)$, assuming that Mr. Claus is compensated for the sleds he makes based on the demand equation given above.
   Solving for p in the demand equation given above, we have $2p = 100 - 4x$, or $p = 50 - 2x$
   Then $R(x) = (\text{price})(\text{quantity}) = (50-2x)(x) = 50x - 2x^2$.

   (c) Find the daily profit function, $P(x)$.
   Recall that $P(x) = R(x) - C(x) = 50x - 2x^2 - (25x + 50) = -2x^2 + 25x - 50$.

   (d) Find the marginal profit function.
   $P'(x) = -4x + 25$

   (e) Find the maximum profit that S. Claus can make each day, assuming that his workshop can produce at most 10 sleds per day (they have to spend time making other types of toys, you know!)
   We wish to find the absoluate maximum of $P(x)$. We'll start by finding the critical points of $P'(x)$:
   If $-4x + 25 = 0$, $25 = 4x$, so $x = 6.25$. Next, we check to see what the value of $P(x)$ is at our critical point and at our endpoints. Notice that at least 0 sleds are made, and at most 10 sleds are made. Notice that $P(0) = -50$, $P(6.25) = -2(6.25)^2 + 25(6.25) - 50 = 28.125$, and $P(10) = -2(10)^2 + 25(10) - 50 = 0$.
   Therefore, the profit function is maximized when $x = 6.25$. But wait, this does not quite answer the question. It is not possible to make 6.25 sleds (who would want to buy 6.25 sleds?!?). To amke sense of this, we really need to check to see which is better, 6 sleds, or 7 sleds, since the max occurs between these two values.
   $P(6) = -2(6)^2 + 25(6) - 50 = 28$, while $P(7) = -2(7)^2 + 25(7) - 50 = 27$. Therefore a maximum profit of \$28 per day is made when 6 sleds are produced.

3. Evaluate the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{x^2 - x - 2}{x^2 - 4} = \frac{0^2 - 0 - 2}{0^2 - 4} = \frac{-2}{-4} = \frac{1}{2}$

   (b) $\displaystyle\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} = \frac{(x-2)(x+1)}{(x-2)(x+2)} = \frac{(x+1)}{(x+2)} = \frac{2+1}{2+2} = \frac{3}{4}$

   (c) $\displaystyle\lim_{x\to \infty} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x\to\infty} \frac{x^2}{x^2} = 1$

4. Use the limit definition of the derivative to compute the derivative function $f'(x)$ if $f(x) = x^2+3x-2$. You must show all work to receive credit for this problem.

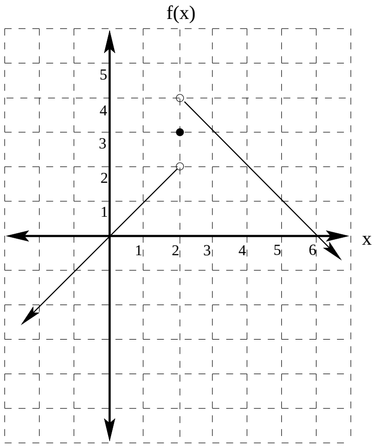
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+3(x+h)-2-(x^2+3x-2)}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2+3x+3h-2-x^2-3x+2}{h} = \lim_{h\to 0} \frac{2xh+h^2+3h}{h} = \lim_{h\to 0} 2x+h+3 = 2x+3$$

5. Given the function:

$$f(x) = \begin{cases} x & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 6-x & \text{if } x > 2 \end{cases}$$

(a) Graph $f(x)$.



(b) Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.
Based on the graph above, $\lim_{x\to 2^-} f(x) = 2$, and $\lim_{x\to 2^+} f(x) = 4$

(c) Is $f(x)$ continuous at $x = 1$? Justify your answer. Yes, when $x = 1$, the two sides limit exists and equals 1, and the function value is also 1. Since function value is defined, the two-sided limit exists, and these two values agree, $f(x)$ is contijnuous at $x = 1$.

6. Use properties of logarithms to expand the following expression:

$$\ln\left(\frac{x^3 z}{\sqrt[5]{xy}}\right) = \ln(x^3 z) - \ln\left((xy)^{\frac{1}{5}}\right) = \ln(x^3) + \ln z - \frac{1}{5}\ln(xy) = 3\ln x + \ln z - \frac{1}{5}\ln x - \frac{1}{5}\ln y = \frac{14}{15}\ln x + \ln z - \frac{1}{5}\ln y$$

7. Find the exact value of each of the following:

(a) $\log_3(\frac{1}{27}) = -3$

(b) $\log_9(\frac{1}{27}) = -\frac{3}{2}$

(c) $\log_{17}(1) = 0$

(d) $5^{\log_5(472)} = 472$

8. Find the interest rate needed for an investment of \$5,000 to triple in 10 years if the interest is compounded continuously.
Using $A = Pe^{rt}$ with $P = 5,000$, $A = 15,000$, adn $t = 10$, we have $15,000 = 5,000e^{10r}$.
Then $3 = e^{10r}$, or $\ln 3 = 10r$. Thus $r = \frac{\ln 3}{10} \approx .1099$, or about 10.99%.

9. Find the derivative of each of the following functions. You **do not** have to use the limit definition, and you **do not** need to simplify your answers.

(a) $f(x) = 5x^4 - 3\sqrt{x} + \ln(2x-3) = 5x^4 - 3x^{\frac{1}{2}} + \ln(2x-3)$
$f'(x) = 20x^3 - \frac{3}{2}x^{-\frac{1}{2}} + \frac{2}{2x-3}$

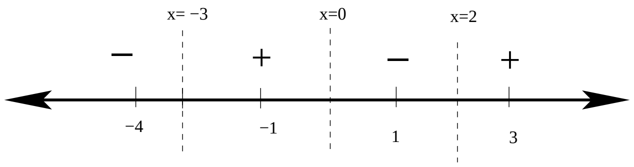
(b) $f(x) = (x^2+2)e^{2-3x}$
Using the product rule, $f'(x) = (2x)e^{2-3x} + (x^2+2)e^{2-3x}(-3) = 2xe^{2-3x} + (-3x^2-6)e^{2-3x}$

(c) $f(x) = \dfrac{2x-5}{x^2+1}$
By the quotient rule, $f'(x) = \frac{(x^2+1)(2)-(2x-5)(2x)}{(x^2+1)^2}$

(d) $f(x) = (4x^4-7x+5)^{\frac{3}{2}}$
$f'(x) = \frac{3}{2}(4x^4-7x+5)^{\frac{1}{2}}(16x^3-7)$

10. Let $f(x) = \frac{3}{4}x^4 + x^3 - 9x^2 + 12$

(a) Find the intervals where $f(x)$ is increasing and those where $f(x)$ is decreasing.
To find these intervals, we must analyze the first derivative. Notice that $f'(x) = 3x^3 + 3x^2 - 18x$.
To find the critical points of $f(x)$, we factor $f'(x)$ and set it equal to zero:
$3x^3 + 3x^2 - 18x = 3x(x^2 + x - 6) = 3x(x + 3)(x - 2) = 0$, therefore, $x = 0$, $x = -3$ and $x = 2$ are our critical points. We than do sign analysis:



From this, we see that $f(x)$ is increasing on $(-3, 0) \cup (2, \infty)$ and decreasing on $(-\infty, -3) \cup (0, 2)$

(b) Find and classify the relative extrema of $f(x)$.
$(-3, f(-3)) = (-3, -35.25)$ is a relative minmum since the function goes from decreasing to increasing at $x = -3$.
$(0, f(0)) = (0, 12)$ is a relative maximum since the function goes from increasing to decreasing at $x = 0$.
$(2, f(2)) = (2, -4)$ is a relative minmum since the function goes from decreasing to increasing at $x = 2$.

(c) Find the equation for the tangent line to $f(x)$ when $x = 2$.
$f'(2) = 3(2)^3 + 3(2)^2 - 18(2) = 0$, so $m = 0$
$f(2) = -4$, so $P = (2, -4)$. Thus the line is given by $y + 4 = 0(x - 2)$, of $y = -4$.

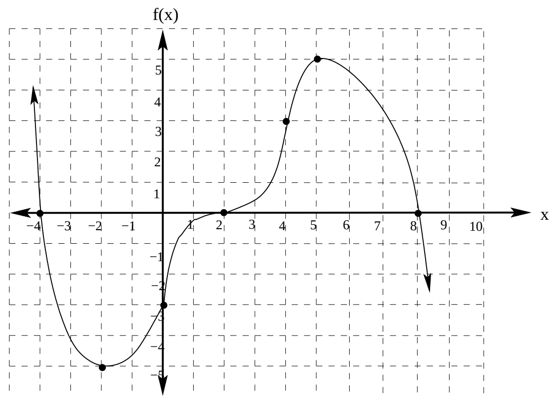
11. Sketch the graph of a function that satisfies the following conditions:

Domain: $(-\infty, \infty)$, x-intercepts: $(-4, 0), (2, 0), (8, 0)$, y-intercept: $(0, -3)$

Increasing on: $(-2, 5)$, Decreasing on: $(-\infty, -2) \cup (5, \infty)$

Concave up on: $(-\infty, 0) \cup (2, 4)$, Concave down on: $(0, 2) \cup (4, \infty)$

Local Max: $(5, 5)$, Local Min: $(-2, -5)$, Inflection Points: $(0, -3), (2, 0), (4, 3)$



12. If the marginal cost of a manufactoring process is known to be $45x^2 + 32x - 700$, and the fixed costs are \$2000, find the cost function $C(x)$.

Since $C'(x) = 45x^2 + 32x - 700$, antidifferentiating, $C(x) = 15x^3 + 16x^2 - 700x + C$. Since the fixed costs are the costs incurred even when nothing is produced, we know that $C(0) = 2000$. Therefore, $C = 2000$, and $C(x) = 15x^3 + 16x^2 - 700x + 2000$.

13. Evaluate the following integrals:

(a) $\int 6\sqrt{x} - 4x^{-1}\ dx = \int 6x^{\frac{1}{2}} - 4x^{-1}\ dx = (6)\frac{2}{3}x^{\frac{3}{2}} - 4\ln|x| + C = 4x^{\frac{3}{2}} - 4\ln|x| + C$

(b) $\int_0^4 2e^{2x} + x^{\frac{1}{2}}\ dx = e^{2x} + \frac{2}{3}x^{\frac{3}{2}}|_0^4 = [e^{2(4)} + \frac{2}{3}(4)^{\frac{3}{2}}] - [e^0 + \frac{2}{3}(0)] = e^8 + \frac{2}{3}(2)^3 - 1 = e^8 + \frac{13}{3} \approx 2985.29$

14. Find the average value of $f(x) = x^3 - \frac{1}{x^2}$ for $-2 \le x \le 2$.

Avg value $= \frac{1}{b-a}\int_a^b f(x)\ dx = \frac{1}{4}\int_{-2}^{2} x^3 - x^{-2}\ dx = \frac{1}{4}\left[\frac{1}{4}x^4 + x^{-1}|_{-2}^{2}\right] = \frac{1}{4}\left[\left(\frac{16}{4} + \frac{1}{2}\right) - \left(\frac{16}{4} - \frac{1}{2}\right)\right] = \frac{1}{4}[1] = \frac{1}{4}.$

15. Given that:

$$A = \begin{bmatrix} 1 & -5 \\ 3 & 0 \end{bmatrix} B = \begin{bmatrix} 2 & -3 \\ 1 & 7 \end{bmatrix} C = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 4 \end{bmatrix} D = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{5} & \frac{1}{15} \end{bmatrix}$$

(a) Find $2A - B$

$$= \begin{bmatrix} 2 & -10 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 5 & -7 \end{bmatrix}$$

(b) Find BC

$$= \begin{bmatrix} 2 & -3 \\ 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} (2-6) & (6+0) & (-4-12) \\ (1+14) & (3+0) & (-2+28) \end{bmatrix} = \begin{bmatrix} -4 & 6 & -16 \\ 15 & 3 & 26 \end{bmatrix}$$

(c) Prove that $DA = AD$.

$$DA = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{5} & \frac{1}{15} \end{bmatrix} \cdot \begin{bmatrix} 1 & -5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} (0+1) & (0+0) \\ (-\frac{1}{5}+\frac{1}{5}) & (1+0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AD = \begin{bmatrix} 1 & -5 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{5} & \frac{1}{15} \end{bmatrix} = \begin{bmatrix} (0+1) & (\frac{1}{3}-\frac{1}{3}) \\ (0+0) & (1+0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

16. Use matrices to solve:

$$\begin{cases} x + 3y + z = 3 \\ 3x + 8y + 3z = 7 \\ 2x - 3y + z = -10 \end{cases}$$

We will solve this system by changing to matrix form and transforming the matrix form of this system:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 3 & 8 & 3 & 7 \\ 2 & -3 & 1 & -10 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & -1 & 0 & -2 \\ 2 & -3 & 1 & -10 \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & -9 & -1 & -16 \end{array}\right]$$

$$\xrightarrow{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & -9 & -1 & -16 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & -9 & -1 & -16 \end{array}\right] \xrightarrow{R_3+9R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therfore, $x = -1, y = 2$, and $z = -2$ is the unique solution to this system of linear equations.

17. The owner of a luxury yacht that sails among the Greek islands charges \$600 per person per day if exactly 20 people sign up for the cruise. However, if more than 20 people sign up for the cruise (up to the maximum capacity of 90 people), then the fare for **every** passenger is reduced by \$4 per day for each passenger in excess of 20. The cruise will not run if fewer than 20 passengers sign up. Find the number of passengers that maximizes the daily revenue for the cruise. Also find the maximum revenue, and the price per person when revenue is maximized.

We acutally have more than case to look at here. When $x < 20$, $R(x) = 0$, since the yacht won't go with less than 20 people. When $x = 20$, then $R(x) = (20)(600) = \$12,000$

For $x > 20$, the fare is reduced by \$4 for each passenger in excess of 20, so we have the following function:

$R(x) = (\text{price})(\text{quantity}) = (x)(600 - 4(\#\text{ of passengers beyond } 20) = (x)(600 - 4(x-20)) = x(600 - 4x + 80) = 600x - 4x^2 + 80x = -4x^2 + 680x$.

Notice that the maximum capacity of the tacht is 90 passengers.

So, we need to maximize $R(x) = -4x^2 + 680x$ on the inteval $(20, 90]$.

To find the critical points, we look at $R'(x) = -8x + 680 = 0$, or $680 = 8x$, so $x = 85$.

We already know $R(0)$, and $R(20$ from above, so we now need to find $R(85)$, and $R(90)$, the value of the revenue function at the critical point and at the right endpoint of the interval.

$R(85) = -4(85)^2 + 680(85) = \$28,900$

$R(90) = -4(90)^2 + 680(90) = \$28,800$

Therefore, revenue is maximized when $x = 85$

The fare when there are 85 passengers is $600 - 65(4) = \$340$ per person.