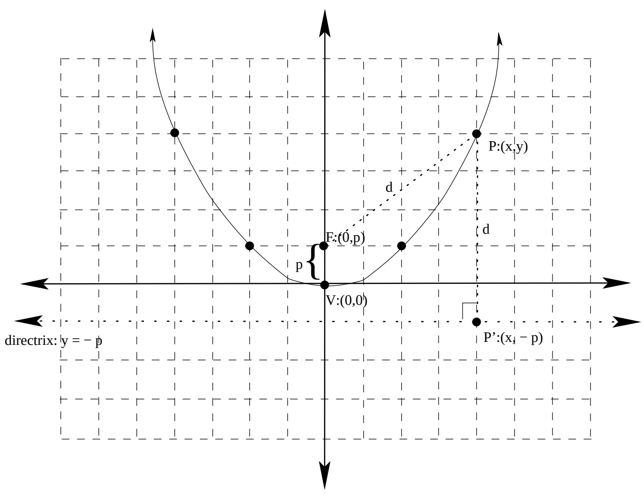
Math 261
Conic Sections Handout
04/22/2007

A Guide to Conic Sections

A. Parabolas

Geometric Definition: A **parabola** is the set of all points in a plane equidistant from a fixed point F (the **focus**) and a fixed line ℓ (the **directrix**) in the plane.

- The **axis** of a parabola is the line through F perpendicular to the directrix ℓ.
- The **vertex** of a parabola if the point V on the axis which is halfway between the focus F and the line ℓ.



Some Useful Formulas:
If $V : (0, 0)$

- The general form of such a parabola is: $y = ax^2$ or $x = ay^2$.
- Up/Down parabolas have equation: $x^2 = 4py$ or $y = \frac{1}{4p}x^2$
- Left/Right parabolas have equation: $y^2 = 4px$ or $x = \frac{1}{4p}y^2$

If $V : (h, k)$

- The general form of such a parabola is: $y = ax^2 + bx + c$ or $x = ay^2 + by + c$.
- Up/Down parabolas have equation: $(x - h)^2 = 4p(y - k)$
- Left/Right parabolas have equation: $(y - k)^2 = 4p(x - h)$

For any parabola, $p = \frac{1}{4a}$.
For an Up/Down parabola, $h = -\frac{b}{2a}$ and the axis has equation $x = -\frac{b}{2a}$.
For a Left/Right parabola, $k = -\frac{b}{2a}$ and the axis has equation $y = -\frac{b}{2a}$..
Finally, if we consider a parabolic mirror, the focus F of a parabola has interesting properties:

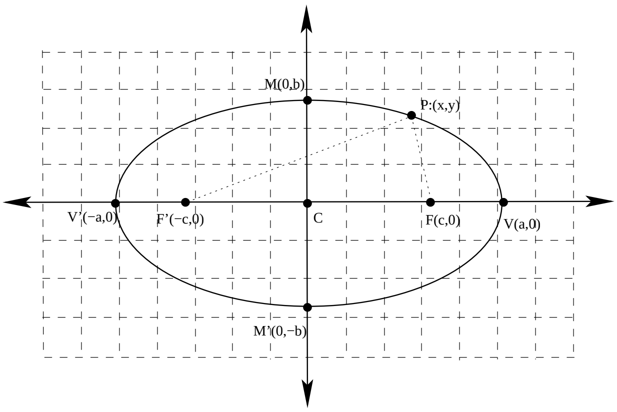
- If a "light source" is placed at F, then all light rays emitted will be reflected so as to travel perpendicular to the axis of the parabola.
- Similarly, a beam of light coming toward a parabolic mirror traveling perpendicular to the axis will be reflected into the focus.

B. Ellipses

Geometric Definition: An **ellipse** is the set of all points in a plane, the *sum* of whose distances from two fixed points F and F' (the **foci**) in the plane is a positive constant.

We can think of constructing an ellipse as follows: Push two thumbtacks in a sheet of paper sitting on top of some cardboard. Then take a piece of string and tie each end onto one of the tacks (with a little bit of slack left over). Finally, take a pencil, pull the string taught around the pencil and trace out a path around the two tacks, guided by the string.

- The midpoint of the line segment connecting the foci is the **center** of the ellipse.
- The points V and V' on the ellipse that are on the line determined by F and F' are called the **vertices** of the ellipse.
- The line segment $\overline{VV'}$ is the **major axis** of the ellipse.
- We use M and M' to denote the points on the ellipse that are on the line which is perpendicular to the line determined by F and F'.
- The line segment $\overline{MM'}$ is the **minor axis** of the ellipse.
- The length of the major axis is denoted by $2a$, and the length of the minor axis is denoted by $2b$.



Some Useful Formulas:

If $C : (0, 0)$

- If the major axis is horizontal, the equation of an ellipse has the form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the ellipse has vertices $(\pm a, 0)$, minor axis endpoints $(0, \pm b)$, and foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$.
- If the major axis is vertical, the equation of an ellipse has the form: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, and the ellipse has vertices $(0, \pm a)$, minor axis endpoints $(\pm b, 0)$, and foci $(0, \pm c)$, where $c^2 = a^2 - b^2$.
- The *eccentricity* e of an ellipse is given by $e = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a}$. Notice that $0 < e < 1$ for any ellipse. The eccentricity can be thought of as a measure of how close an ellipse is to being circular. If $e \approx 0$ then the ellipse is nearly circular, while if $e \approx 1$, then the ellipse is almost "flat".
- The "reflective property" of ellipses is that if a wave or ray of light emanates from one focus of an ellipse, it will pass through the other focus.

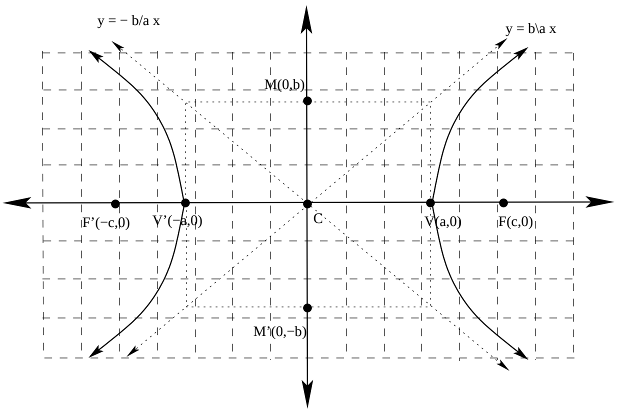
If $C : (h, k)$

- If the major axis is horizontal, the equation of an ellipse has the form: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, and the ellipse has vertices $(h \pm a, k)$, minor axis endpoints $(h, k \pm b)$, and foci $(h \pm c, 0)$, where $c^2 = a^2 - b^2$.
- If the major axis is vertical, the equation of an ellipse has the form: $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, and the ellipse has vertices $(h, k \pm a)$, minor axis endpoints $(h \pm b, k)$, and foci $(h, k \pm c)$, where $c^2 = a^2 - b^2$.

C. Hyperbolas

Geometric Definition: A **hyperbola** is the set of all points in a plane, the *difference* of whose distances from two fixed points F and F' (the **foci**) in the plane is a positive constant.

- The midpoint of the line segment connecting the foci is the **center** of the hyperbola.
- The points V and V' on the hyperbola that are on the line determined by F and F' are called the **vertices** of the hyperbola.
- The line segment $\overline{VV'}$ is called the **transverse axis** of the hyperbola.
- If we let a be half the distance between the vertices and c be half the distance between the foci, then $c > a$. Let $b^2 = c^2 - a^2$.
- We use W and W' to denote the points on the line perpendicular to the **transverse axis** of the hyperbola and each a distance b from the center of the hyperbola.
- The line segment $\overline{MM'}$ is the **conjugate axis** of the hyperbola.



Some Useful Formulas:
If $C : (0, 0)$

- If the transverse axis is horizontal, the equation of a hyperbola has the form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and the hyperbola has vertices $(\pm a, 0)$, conjugate axis endpoints $(0, \pm b)$, and foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$. Also, the hyperbola has *asymptotes* $y = \pm\frac{b}{a}$.
- If the major axis is vertical, the equation of a hyperbola has the form: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, and the hyperbola has vertices $(0, \pm a)$, conjugate axis endpoints $(\pm b, 0)$, and foci $(0, \pm c)$, where $c^2 = a^2 + b^2$. Also, the hyperbola has *asymptotes* $y = \pm\frac{a}{b}$.

If $C : (h, k)$

- If the transverse axis is horizontal, the equation of a hyperbola has the form: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, and the hyperbola has vertices $(h \pm a, k)$, conjugate axis endpoints $(h, k \pm b)$, and foci $(h \pm c, k)$.
- If the major axis is vertical, the equation of a hyperbola has the form: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, and the hyperbola has vertices $(h, k \pm a)$, conjugate axis endpoints $(h \pm b, k)$, and foci $(h, k \pm c)$, where $c^2 = a^2 + b^2$.