Math 261
Differentiation Rules

Differentiation

A. The Formal Definition

Given a function $f(x)$, $f'(x)$, the **derivative** of $f(x)$, is a function that gives the slope of the tangent line to $f(x)$ at any point x (provided such a slope exists). The function $f'(x)$ is defined as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Suppose $f(x) = 2x^2 - 3x + 7$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 - 3(x+h) + 7 - (2x^2 - 3x + 7)}{h}$

$= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) - 3x - 3h + 7 - (2x^2 - 3x + 7)}{h}$

$= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 7 - 2x^2 + 3x - 7}{h} = \lim_{h\to 0} \frac{4xh + 2h^2 - 3h}{h} = \lim_{h\to 0} 4x + 2h - 3 = 4x - 3$

B. Tangent Lines

The tangent line to a function $f(x)$ when $x = a$ is a line containing the point $(a, f(a))$ with slope equal to the instantaneous rate of change of the function f when $x = a$. To find an equation for the tangent line of a function, we first find the point $P = (a, f(a))$ by evaluating the function f when $x = a$. Then, we find the slope by finding the derivative function $f'(x)$ and then evaluating the derivative function when $x = a$, $m = f'(a)$. Finally, we apply the point/slope formula to the point P and the slope m to find the equation of the line.

Example: If $f(x) = 2x^2 - 3x + 7$, find the tangent line to $f(x)$ when $x = -1$.
First notice that when $x = -1$, $f(-1) = 2(-1)^2 - 3(-1) + 7 = 2 + 3 + 7 = 12$, so the point of tangency is $P = (-1, 12)$.
From above, $f'(x) = 4x - 3$, so $m = f'(-1) = 4(-1) + 3 = -4 - 3 = -7$.
Thus, by point/slope, $y - 12 = -7(x + 1)$, or $y = -7x + 5$.

C. Basic Differentiation Formulas

1. Differentiating Power Functions: $\frac{d}{dx}(x^r) = rx^{r-1}$ **Example:** $\frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{-\frac{1}{3}}$

2. Differentiating a constant: $\frac{d}{dx}c = 0$ for any constant c. **Example:** $\frac{d}{dx}(12) = 0$

3. Constant Multiples: $\frac{d}{dx}(cf(x)) = cf'(x)$ for any constant c. **Example:** $\frac{d}{dx}\left(\frac{2}{3}x^3\right) = (\frac{2}{3})(3)x^2 = 2x^2$

4. Sums and Differences: $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$,

   and $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$ for any functions f and g.

   Example: $\frac{d}{dx}\left(3x^2 - x^{\frac{3}{2}}\right) = 6x - \frac{3}{2}x^{\frac{1}{2}}$

5. Products: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

   Example: $\frac{d}{dx}((3x^3 - 4x + 7)(12x^4 - 13x^3 - 7x + 4))$
   $= (9x^2 - 4)(12x^4 - 13x^3 - 7x + 4) + (3x^3 - 4x + 7)(48x^3 - 39x^2 - 7)$

6. Quotients: $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

   Example: $\frac{d}{dx}(\frac{3x^3 - 4x + 7}{12x^4 - 13x^3 - 7x + 4})$
   $= \frac{(9x^2 - 4)(12x^4 - 13x^3 - 7x + 4) - (3x^3 - 4x + 7)(48x^3 - 39x^2 - 7)}{[12x^4 - 13x^3 - 7x + 4]^2}$