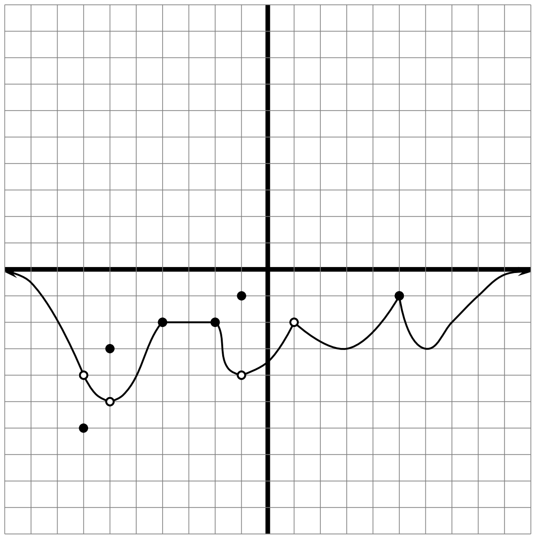
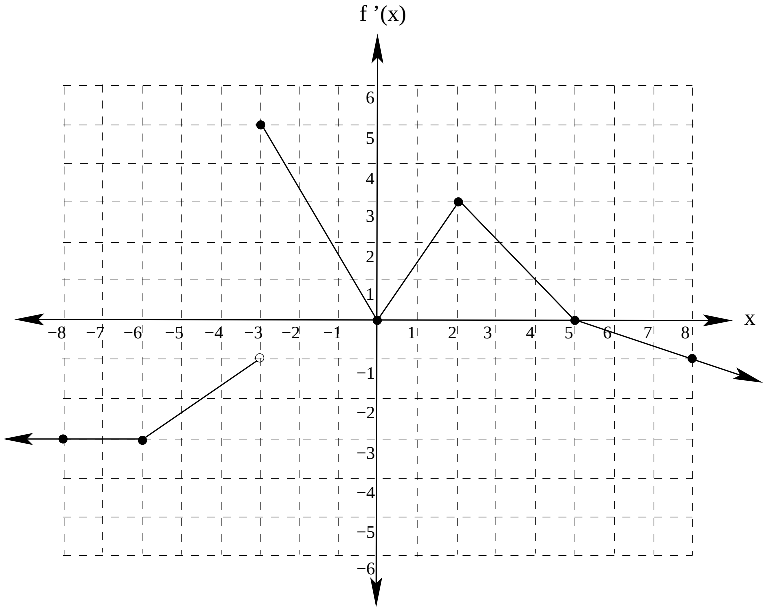
Math 261
Exam 3 - Practice Problems

1. The graph of f is given below. Answer the following questions.



   (a) Find the intervals where f is increasing: $(-6, -4)$, $(-1, 1), (3, 5), (6, \infty)$
   (b) Find the intervals where f is decreasing: $(-\infty, -7)$, $(-7, -6), (-2, -1), (1, 3), (5, 6)$
   (c) Find the intervals where f is constant: $(-4, -2)$
   (d) Find all local maximums: $y = -3, -2, -1$
   (e) Find the location of all local maximums: $x = -6$, $x \in [-4, -2]$, $x = -1, x = 5$
   (f) Find all local minimums: $y = -6, -2, -3$
   (g) Find the location of all local minimums: $x = -7, x \in (-4, -2), x = 3, x = 6$
   (h) Find the absolute maximum and its location, if it exists: absolute max does not exist
   (i) Find the absolute minimum and its location, if it exists: absolute min is -6 at $x = -7$
   (j) Find the absolute maximum on the interval $[3, 8]$, if it exists: absolute max is -1
   (k) Find the absolute minimum on the interval $[3, 8]$, if it exists: absolute min is -3

2. Answer the questions below based on the graph of $f'(x)$ shown here:



(a) Find the intervals where $f(x)$ is decreasing.

Recall that $f(x)$ is decreasing when $f'(x)$ is negative. Therefore, $f(x)$ is decreasing on the intervals:

$(-\infty, -3) \cup (5, \infty)$

(b) Give the x coordinates of the local extrema of f, and state whether each is a local maximum or a local minimum.

$f(x)$ has critical numbers when $f'(x)$ changes signs. When $f'(x)$ changes from negative to positive, f has a local minimum. When $f'(x)$ changes from positive to negative, f has a local maximum. Therefore, f has a local minimum at $x = -3$ and a local maximum at $x = 5$.

Notice that although $f'(0) = 0$, $f'(x)$ does not change signs at that point, so $x = 0$ is a critical number but not a local extremum.

(c) Find the intervals where $f(x)$ is concave down.

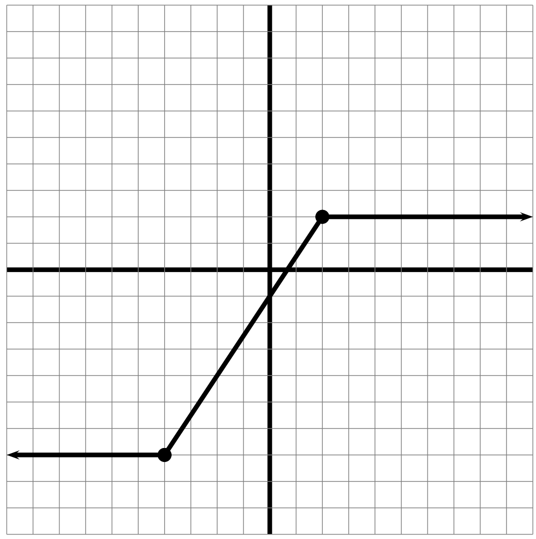
$f(x)$ is concave down when $f'(x)$ is decreasing. Therefore, $f(x)$ is concave down on the intervals:

$(-3, 0) \cup (2, \infty)$

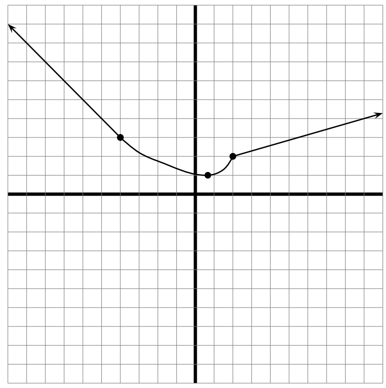
(d) Give the x-coordinates of any inflection points.

f has inflection points wherever $f'(x)$ changes from increasing to decreasing or vice versa. Therefore, f has inflection points at the points: $x = -3$, at $x = 0$, and at $x = 2$.

3. The graph of f'' is shown. Answer the following questions.



(a) Sketch a possible graph of f': (The x-scale is by 1's and the y-scale is by 7's)



(b) Where is f concave up?

$\left(\frac{2}{3}, \infty\right)$ (note: the $\frac{2}{3}$ is exact)

(c) Where is f concave down?

$\left(-\infty, \frac{2}{3}\right)$

(d) Where is f' linear?

f' is linear where its slope is constant, so where f'' is constant. So on the intervals $(-\infty, -4)$ and $(2, \infty)$.

4. Find the absolute maximum and absolute minimum value of each of the following functions in the given interval:

(a) $f(x) = 2x^2 - 7x + 1$ on the interval $[0, 9]$

Notice that $f'(x) = 4x - 7$, so $f(x)$ has one critical point, when $x = \frac{7}{4}$.

Since $f(0) = 1$, $f(9) = 162 - 63 + 1 = 100$, and $f(\frac{7}{4}) = -\frac{41}{8}$, max: 100, min: $-\frac{41}{8}$

(b) $f(x) = x^3 - 9x + 5$ on the interval $[-2, 5]$
Here, $f'(x) = 3x^2 - 9$, so $f(x)$ has critical points when $3x^2 = 9$, or at $x = \pm\sqrt{3}$
Since $f(-2) = -8 + 18 + 5 = 15, f(5) = 125 - 45 + 5 = 85$, $f(-\sqrt{3}) = -3\sqrt{3} + 9\sqrt{3} + 5 = 6\sqrt{3} + 5 \approx 15.3923$, and $f(\sqrt{3}) = 3\sqrt{3} - 9\sqrt{3} + 5 = -6\sqrt{3} + 5 \approx -5.3923$,
max: 85, min: $-6\sqrt{3} + 5$

(c) $f(x) = \dfrac{x+3}{2x-1}$ on the interval $[1, 10]$
First notice that $f(x)$ is continuous except when $x = \frac{1}{2}$, which is outside our interval, so the EVT applies to this function on the interval $[1, 10]$.
Next, $f'(x) = \frac{1(2x-1)-(x+3)2}{(2x-1)^2} = \frac{-7}{(2x-1)^2}$, so the only critical point $x = \frac{1}{2}$, is outside the interval being considered.
Since $f(1) = 4$, and $f(10) = \frac{13}{19}$, max: 4, min: $\dfrac{13}{19}$

(d) $f(x) = \dfrac{x}{1+x^2}$ on the interval $[-2, 4]$
First notice that f is continuous in the interval $[-2, 4]$ (in fact it is continuous everywhere since the denominator is never zero).
Therefore, the absolute extrema of f on the interval $[-2, 4]$ occur either at critical numbers or at endpoints.
Now, $f'(x) = \dfrac{(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. Since the denominator is never zero, all critical numbers occur when $1 - x^2 = 0$, or when $x = \pm 1$.
We now compute values for f at the critical points and both endpoints:
$f(-2) = \frac{-2}{1+4} = -\frac{2}{5}$
$f(1) = \frac{-1}{1+1} = -\frac{1}{2}$ – absolute minimum
$f(-1) = \frac{1}{1+1} = \frac{1}{2}$ – absolute maximum
$f(4) = \frac{4}{1+16} = \frac{4}{17}$

5. For each of the following, assuming that $f(x)$ is both continuous and differentiable everywhere, state what graph feature occurs at $f(2)$ (i.e. there is a local maximum, a local minimum, inflection point, or more information is needed):

(a) $f(2) = -5$, $f'(2) = 0$, and $f''(2) = -1$
Notice that f has a critical point and is concave down when $x = 2$, therefore, this point must be a local maximum.

(b) $f(2) = 7$, $f'(2) = -3$, and $f''(2) = 0$
More information is needed. We can see that f is decreasing and has zero second derivative when $x = 2$. Although we might suspect that this point is an inflection point, we cannot be sure without knowing what the concavity is on either side of $x = 2$. What if the function is linear on on interval containing $x = 2$?

(c) $f(2) = 1$, $f'(2) = 0$, and $f''(2) = 3$
Notice that f has a critical point and is concave up when $x = 2$, therefore, this point must be a local minimum.

6. Sketch the graph of a function $f(x)$ that satisfies the following:

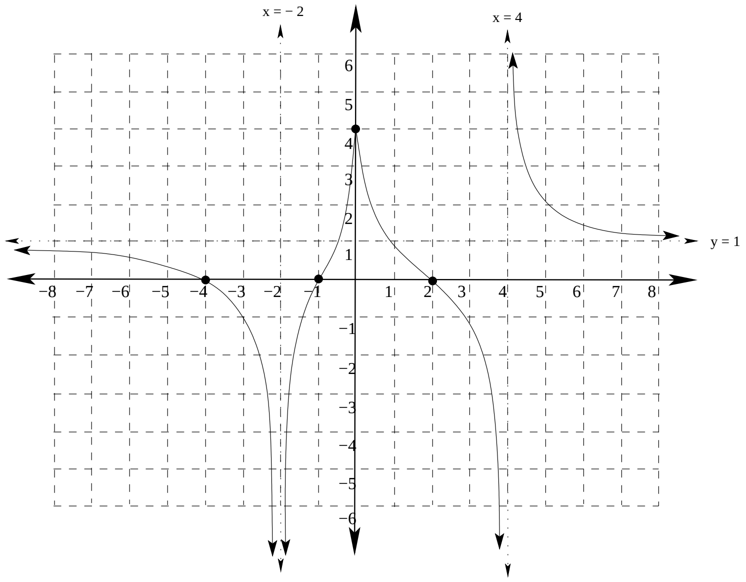
Domain: $(-\infty,-2)\cup(-2,4)\cup(4,\infty)$; x-intercepts: $(-4,0),(-1,0)$, and $(2,0)$; y-intercept: $(0,4)$

Increasing on: $(-2,0)$; Decreasing on: $(-\infty,-2)\cup(0,4)\cup(4,\infty)$

Concave up on: $(-1,0)\cup(0,2)\cup(4,\infty)$; Concave down on: $(-\infty,-2)\cup(-2,-1)\cup(2,4)$

Local Max: $(0,4)$, Local Mins: none

$\lim_{x\to-\infty} f(x)=\lim_{x\to\infty} f(x)=2$; $\lim_{x\to-2^-} f(x)=-\infty$; $\lim_{x\to-2^+} f(x)=-\infty$; $\lim_{x\to4^-} f(x)=-\infty$; $\lim_{x\to4^+} f(x)=\infty$



7. Find any local extrema of the following functions. Classify each as a maximum or a minimum.

(a) $f(x)=x^3-7x^2-5x+6$
$f'(x)=3x^2-14x-5$, so f has critical points when $(3x+1)(x-5)=0$, or when $x=-\frac{1}{3}$ and $x=5$.
$f''(x)=6x-14$, so f is concave up when $x=5$ and concave down when $x=-\frac{1}{3}$.
Hence there is a local maximum of $\dfrac{185}{27}$ when $x=-\frac{1}{3}$, and a local minimum of -69 when $x=5$.

(b) $f(x)=x^3\sqrt{x}-14x^2+10=x^{\frac{7}{2}}-14x^2+10$
$f'(x)=\frac{7}{2}x^{\frac{5}{2}}-28x$, so f has critical points when $x(\frac{7}{2}x^{\frac{3}{2}}-28)=0$. That is, when $x=0$ or when $\frac{7}{2}x^{\frac{3}{2}}=28$, or $x^{\frac{3}{2}}=8$, so when $x=4$.
$f''(x)=\frac{35}{4}x^{\frac{3}{2}}-28$, so f is concave down when $x=0$ and concave up when $x=4$.
Hence there is a local maximum of 10 when $x=0$ and a local minimum of -86 when $x=4$.

8. A particle is moving along a straight line during a sixty second time interval. Every ten seconds the position, velocity, and acceleration were measured and recorded in a table below. Answer each question below and give a justification for your answers.

t (*seconds*)	0	10	20	30	40	50	60
$s(t)$ (*feet*)	0	5	25	40	90	75	25
$v(t)$ (*feet/sec*)	0	2	3	2	1	-1	1
$a(t)$ (*feet/sec*^{2})	10	5	3	1	1	2	5

(a) Is there a time when $s(t) = 20$?
Yes. Notice that $s(10) = 5$ and $s(20) = 25$. Also, $s(t)$ is continuous, so by the Intermediate Value Theorem, f attains every value between 5 and 25. That is, since $s(10) = 5 < 20 < s(20) = 25$, then for some $10 < c < 20$, $s(c) = 20$.

(b) Is there a time when $v(t) = 5$?
Yes. Notice that the average value of $v(t)$ on the interval $[30, 40]$ is $\dfrac{s(40) - s(30)}{40 - 30} = \dfrac{90 - 40}{10} = 5$.
Therefore, by the Mean Value Theorem, for some $30 < c < 40$, $s'(t) = v(t) = 5$.

(c) Is there a time when $a(t) = 0$?
Yes. Since $v(10) = v(30) = 2$, by Rolle's Theorem, for some $10 < c < 30$ $v'(t) = a(t) = 0$.

9. Find a number c in the given interval that satisfies the Mean Value Theorem for the function and interval given, or explain why the Mean Value Theorem does not apply.

(a) $f(x) = x^3 - x$ on $[-1, 1]$
$f'(x) = 3x^2 - 1$, $m_{sec} = \frac{f(1)-f(-1)}{2} = 0$
So we want to find the c-values where $3c^2 - 1 = 0$. Hence, $c = \pm\dfrac{1}{\sqrt{3}}$

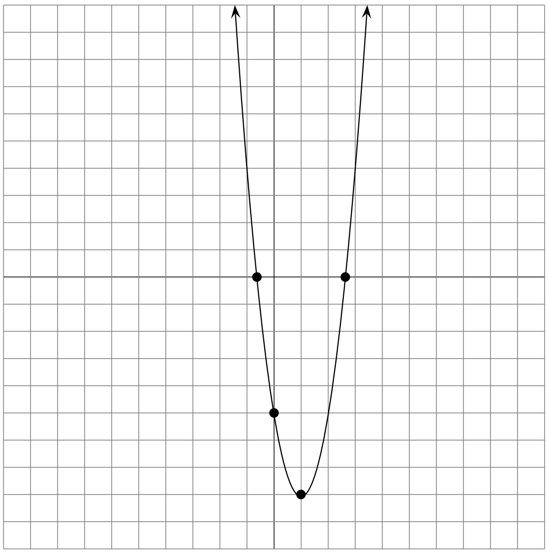
(b) $f(x) = \dfrac{x+1}{x-1}$ on $[2, 5]$
$f'(x) = \dfrac{-2}{(x-1)^2}$. $m_{sec} = \frac{f(5)-f(2)}{3} = \frac{\frac{3}{2}-3}{3} = -\frac{1}{2}$
So we want to find the c-values where $\dfrac{-2}{(c-1)^2} = -\dfrac{1}{2}$, or $(c-1)^2 = 4$
Hence, $c - 1 = \pm 2$, so $c = 3$ ($c = -1$ is outside our interval).

(c) $f(x) = \dfrac{x+1}{x-1}$ on $[-1, 2]$
This function is not continuous on $[-1, 2]$, so the MVT does not apply.

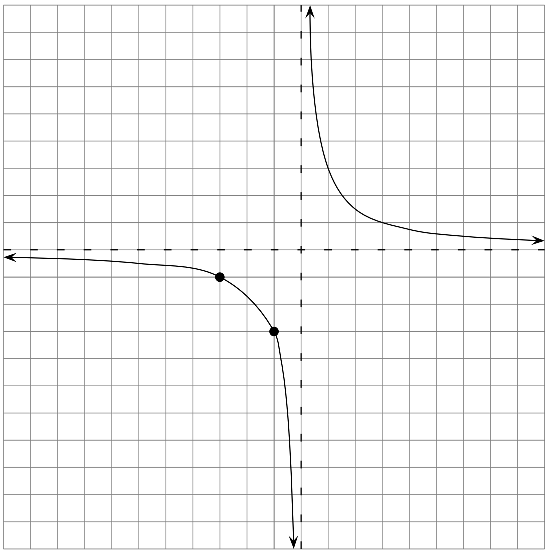
(d) $h(\theta) = \sin\theta + \cos\theta$ on $[0, 2\pi]$
$h'(\theta) = \cos\theta - \sin\theta$. $m_{sec} = \frac{h(0)-h(2\pi)}{2\pi} = 0$
So we want to find the c-values where $\cos c - \sin c = 0$, or $\cos c = \sin c$
Hence, $c = \frac{\pi}{4}$ or $c = \dfrac{5\pi}{4}$

10. For each of the following functions:
- Find the intercepts of $f(x)$.
- Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.
- Find and classify all local extrema of $f(x)$.
- Find the intervals where $f(x)$ is concave up and the intervals where $f(x)$ is concave down.
- Find any inflection points of $f(x)$.
- Sketch the graph of $f(x)$ accurately enough to show all relative extrema and inflection points.

(a) $f(x) = 3x^2 - 6x - 5$
- Domain $(-\infty, \infty)$
- y-intercept $(0, -5)$
- x-intercept $\left(\frac{3+2\sqrt{6}}{3}, 0\right), \left(\frac{3-2\sqrt{6}}{3}, 0\right)$
- Decreasing $(-\infty, 1)$
- Increasing $(1, \infty)$
- Minimum at $(1, -8)$
- Concave up $(-\infty, \infty)$



(b) $f(x) = \dfrac{x+2}{x-1}$
- Domain $\{x | x \neq 1\}$
- y-intercept $(0, -2)$
- x-intercept $(-2, 0)$
- vertical asymptote $x = 1$
- horizontal asymptote $y = 1$
- Decreasing $(-\infty, 1), (1, \infty)$
- Concave up $(-\infty, 1)$
- Concave down $(1, \infty)$

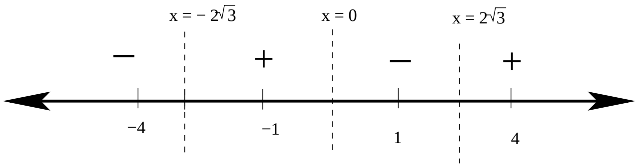


(c) $f(x) = x^4 - 24x^2$.

Intercepts: Notice that $f(0) = 0$, so the y-intercept is $(0, 0)$. Similarly, considering $f(x) = 0$, or $x^4 - 24x^2 = x^2(x^2 - 24) = 0$, we see either $x^2 = 0$, so $x = 0$, or $x^2 = 24$, so $x = \pm\sqrt{24} = \pm 2\sqrt{6}(\approx \pm 4.89)$. Therefore, the x-intercepts are at $(-2\sqrt{6}, 0)$, $(0, 0)$, and $(2\sqrt{6}, 0)$

Incr/Decr: Next, since $f'(x) = 4x^3 - 48x$, the critical numbers of f occur when $4x^3 - 48x = 4x(x^2 - 12) = 0$. That is, when $x = 0$ or $x = \pm\sqrt{12} = \pm 2\sqrt{3}(\approx 3.46)$

testing signs on the regions between these critical numbers:



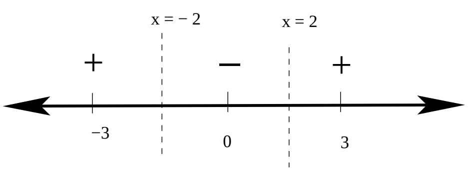
We see that $f(x)$ is decreasing on the intervals: $(-\infty, -2\sqrt{3}) \cup (0, \sqrt{3})$

while $f(x)$ is increasing on the intervals: $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$

Local Extrema: Once again referring to the sign diagram above, there are local extrema at all three critical numbers where $f'(x)$ changes signs. There are local minima where $f'(x)$ changes from negative to positive, that is, at $x = \pm 2\sqrt{3}$ and a local maximum where $f'(x)$ changes from positive to negative, that is, at $x = 0$. Evaluating f on these x-values gives the coordinates of the local extrema: $f(0) = 0$, and $f(2\sqrt{6}) = f(-2\sqrt{6}) = 144 - 24(12) = -144$, so $(0, 0)$, and $\pm 2\sqrt{6}, -144$ are the coordinates of the local extrema.

Concavity: Next, since $f''(x) = 12x^2 - 48$, the critical numbers of f occur when $12x^2 - 48 = 0$. That is, when $12x^2 = 48$ or $x^2 = 4$, so at $x = \pm 2$

testing signs on the regions between these values:

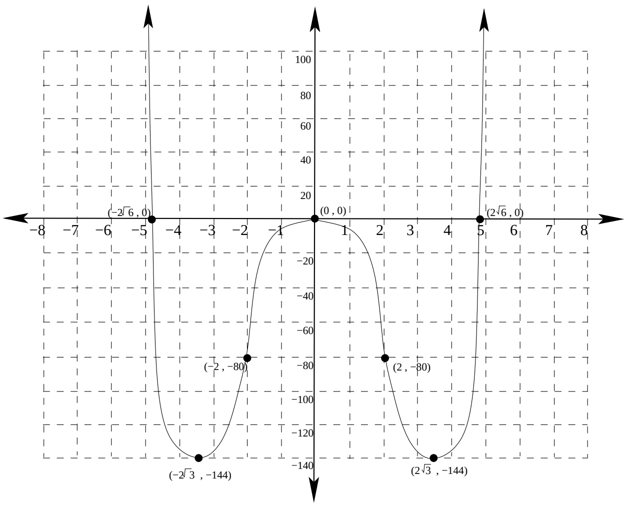


We see that $f(x)$ is concave up on the intervals: $(-\infty, -2) \cup (2, \infty)$

while $f(x)$ is concave down on the interval: $(-2, 2)$

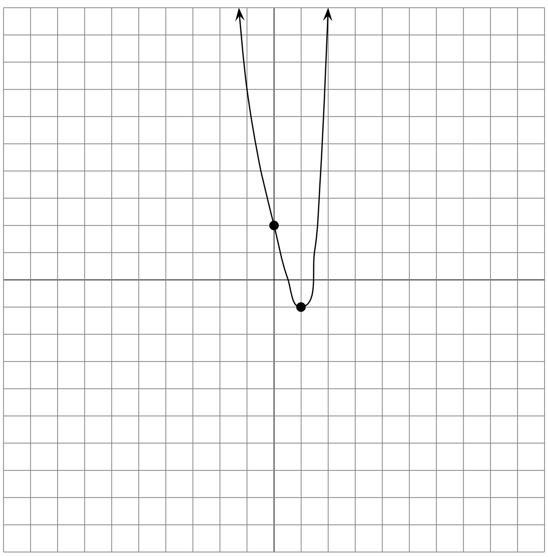
Inflection pts: Since the second derivative changes signs twice (at $x = \pm 2$, there are two inflection points. We find the y coordinates by evaluating $f(2) = f(-2) = (-2)^4 - 24(-2)^2 =$

$16 - 24(4) = -80$. Hence the inflection points are at $(-2, -80)$ and at $(2, -80)$. Combining all of this information, we obtain the following graph:



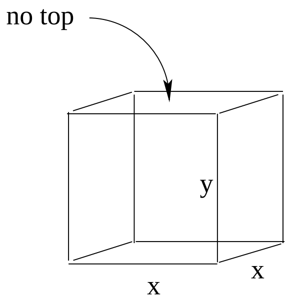
(d) $f(x) = x^4 - 4x + 2$

- Domain $(-\infty, \infty)$
- y-intercept 2
- no asymptotes
- decreasing $(-\infty, 1)$
- increasing $(1, \infty)$
- the point $(1, -1)$ is a local minimum
- concave up on $(-\infty, \infty)$
- no points of inflection



11. Solve the following Optimization problems:

(a) A box with square base and open top is to have a volume of 4 cubic feet. Find the dimensions of the box that will minimize the surface area of the box.



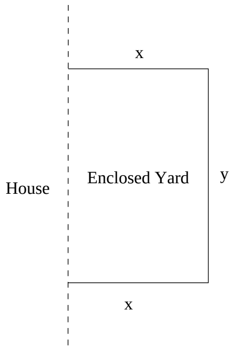
Let x be the length of a side of the square base of the box, and y the height of the box. Then the volume of the box is given by $V = x^2y = 4$ cubic feet. Hence $y = \frac{4}{x^2}$. Next, since the box has no top, the surface area of the box is given by the area of the base plus the sum of the area of the four sides. That is, $A = x^2 + 4xy$. Substituting, $A(x) = x^2 + (4x) \cdot \frac{4}{x^2} = x^2 + \frac{16}{x}$.

Now, we find the critical numbers for this function: $A'(x) = 2x - 16x^{-2} = 2x - \frac{16}{x^2} = \frac{2x^3-16}{x^2}$. Notice that $x = 0$ does not make sense for a box, so we only consider $2x^3 - 16 = 0$, in which case, $2x^3 = 16$, $x^3 = 8$, so $x = 2$ is a critical number.

Notice that $A'(1) = -14 < 0$, and $A'(3) = \frac{54-16}{9} = \frac{38}{9} > 0$, so there is a local minimum at $x = 2$. Since x could hypothetically be as large as we wish, there are no other boundary points. Hence the surface area of the box must be minimized when $x = 2$ and $y = \frac{4}{2^2} = 1$.

Thus the dimensions of the box with minimal surface area and volume 4 cubic feet is 2 feet long, 2 feet high, and 1 foot tall.

(b) A field that is to be used for a play area for a large dog is to be fenced in. The field is adjacent to a house, and the side with the house will need no fence. If 500 feet of fence is available, find the dimensions of the field that maximum the area in which the dog can play in.



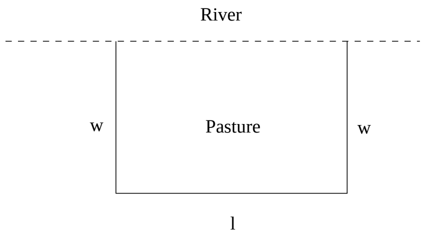
Let y be the length of the side parallel to the house and x the length of the sides perpendicular to the house. Then $P = 2x+y = 500$ feet, thus $y = 500-2x$. Also, $A = xy$, so $A(x) = 500x-2x^2$.

Therefore, $A'(x) = 500 - 4x$, so we have once critical point where $500 = 4x$, or $x = 125$.

Since $A''(x) = -4$, this is a local maximum. This must the the absolute maximum of all the values that "make sense", since the boundary values of $x = 0$ and $x = 250$ lead to yards with zero area.

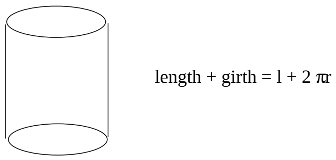
Therefore, the dimensions of the largest yard are 125×250 feet, where the longer side is parallel to the house.

(c) The owner of a ranch has 3000 yards of material to enclose a pasture along a straight stretch of river. If fencing is not required along the river, find the dimensions of the pasture of largest area that he can enclose.



Let l be the width of the fence and w the width. Then $2w + l = 3000$, and $A = l \cdot w = w(3000-2w) = 3000w-2w^2$. Therefore, $A'(w) = 3000-4w$, which as a critical when $3000 = 4w$, or when $w = 750$. Since $A''(w) = -4$, we see by the Second Derivative test that this is a local maximum. Finally, since $0 \leq w \leq 1500$, and the boundary values give pastures with zero area, the pasture of largest area has width 750 yards and length 1500 yards.

(d) Postal regulations require a parcel sent through US mail to have a combined length and girth of at most 108 inches. Find the dimensions of the cylindrical package of greatest volume that may be sent through the mail.



Notice that the girth of a cylindrical package is $2\pi r$. Therefore, $108 = l + 2\pi r$, or $l = 108 - 2\pi r$. The volume of a cylinder is $v = \pi r^2 l = \pi r^2(108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3$.
Then $V'(r) = 216\pi r - 6\pi^2 r^2 = -6\pi r(\pi r - 36)$. Critical numbers: $r = 0$ and $r = \frac{36}{\pi}$.
Notice that $V''(r) = 216\pi - 12\pi^2 r$, which is positive when $r = 0$, and negative when $r = \frac{36}{\pi}$. Therefore, the maximum volume parcel is one with a radius of $\frac{36}{\pi}$ inches and length $108 - 2\pi\frac{36}{\pi} = 108 - 72 = 36$ inches.

(e) The owner of a yacht charges \$600 per person if exactly 20 people sign up for a cruise. If more than 20 sign up for the cruise (up to a maximum capacity of 90), then the fare for each passenger is discounted by \$4 for each additional passenger beyond 20. Assuming that at least 20 people sign up, determine the number or passengers that will maximize the revenue for the cruise and find the fare for each passenger.

Let x be the number of people *beyond* the original 20 people who sign up for the cruise. Then the total number of people who go is $20 + x$, where $0 \leq x \leq 70$, and the price per person for each passenger is $600 - 4x$. Hence the revenue for the cruise is $R(x) = (20 + x)(600 - 4x) = -4x^2 + 520x + 12000$.

Therefore, $R'(x) = -8x + 520$, which has a critical number when $-8x + 520 = 0$, or when $520 = 8x$, and $x = 65$. Moreover, $R''(x) = -8$, so this is a local minimum.

Finally, we can easily check to see that:

$R(0) = (20)(600) = \$12,000$

$R(65) = (85)(600 - 4 \cdot 65) = \$28,900$

$R(70) = (90)(600 - 4 \cdot 70) = \$27,200$

Hence the revenue for the cruise is maximized when $x = 65$. That is, when 85 people take the cruise. The fare for each passenger here is $600 - 4 \cdot 65) = \$340$.