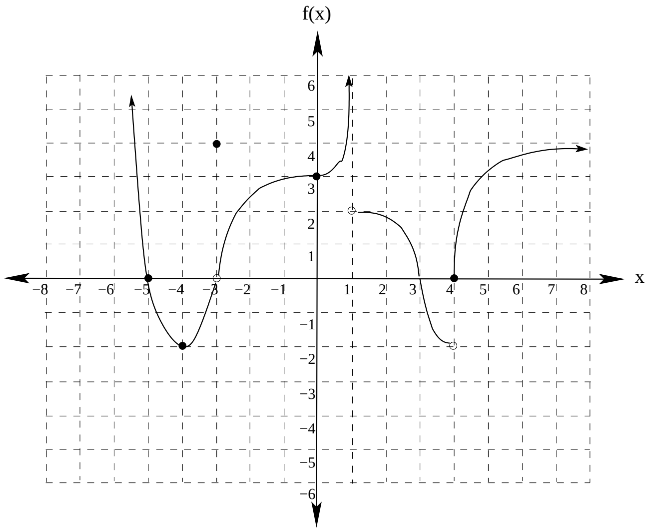
Math 261
Final Exam - Practice Problems

1. A function f is graphed below.



(a) Find $f(0)$, $f(-2)$, $f(1)$, and $f(4)$

(b) Find the domain and range of f

(c) Find the intervals where $f'(x)$ is positive

(d) Find the intervals where $f''(x)$ is negative.

(e) Find $\lim\limits_{x\to -2} f(x)$

(f) find $\lim\limits_{x\to 4^-} f(x)$ and $\lim\limits_{x\to 4^+} f(x)$

(g) find $\lim\limits_{x\to -1^-} f(x)$ and $\lim\limits_{x\to -1^+} f(x)$

(h) find $\lim\limits_{x\to -\infty} f(x)$ and $\lim\limits_{x\to \infty} f(x)$

(i) find the points where $f(x)$ is discontinuous, and classify each point of discontinuity.

2. Evaluate the following limits:

(a) $\lim\limits_{x\to 1} \dfrac{2x^2-5x-3}{3x^2-4x-15}$

(b) $\lim\limits_{x\to 3} \dfrac{2x^2-5x-3}{3x^2-4x-15}$

(c) $\lim\limits_{x\to 2} \sqrt{2x-4}$

(d) $\lim\limits_{x\to \pi} \cos x$

(e) $\lim\limits_{x\to \infty} \cos x$

(f) $\lim\limits_{x\to \infty} \dfrac{2x^2-5x-3}{3x^2-4x-15}$

(g) $\lim\limits_{x\to \infty} \dfrac{2x^2-5x-3}{3x^3-4x-15}$

3. Given the function
$$f(x) = \begin{cases} x+5 & \text{if } x \le -2 \\ x^2-1 & \text{if } |x| < 2 \\ 4-x & \text{if } x \ge 2 \end{cases}$$

   (a) Graph $f(x)$.
   (b) Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to -2} f(x)$
   (c) Is $f(x)$ continuous at $x = 1$? Justify your answer.

4. Give the formal ϵ - δ definition of the limit of a function as presented in class. Then draw a diagram illustrating the definition. Finally, write the definition informally in your own words.

5. Given that $f(x) = 3x^2 - 1$, $\lim_{x\to 1} f(x) = 2$, and $\epsilon = .01$, find the largest δ such that if $0 < |x-1| < \delta$, then $|f(x) - 2| < \epsilon$.

6. Use the formal definition of a limit to prove that $\lim_{x\to 2} 5 - 2x = 1$.

7. Let $f(x) = \dfrac{2x^2-4x}{x^2-x-2}$.

   (a) Find the values of x at which f is discontinuous.
   (b) Find all vertical and horizontal asymptotes of f.

8. Find the x values at which $f(x) = \sqrt{3-2x} + \dfrac{1}{\sqrt{2x+5}}$ is continuous.

9. (a) Use the Intermediate Value Theorem to show $f(x) = 2x^3 + 3x - 4$ has a root between 0 and 1.
   (b) Use Newton's method to approximate this root to 4 decimal places.

10. Find the derivative $y' = \frac{dy}{dx}$ for each of the following:

   (a) $y = \pi^3 + \pi^2 x - \pi x^3 + x^\pi$
   (b) $y = \cos(3x) + \sin(3x)$
   (c) $x^4 + \cos(x^4)$
   (d) $y = \sqrt{x}\tan x$
   (e) $y = \sec^3(x^3)$
   (f) $y = \frac{3-x}{x^2+1}$
   (g) $y = \dfrac{x^2\cos x}{x + \sin(3-2x)}$
   (h) $\sin^2(\tan(x^3-5))$
   (i) $x^2 - 3xy + y^2 = 0$
   (j) $2x^2y - 5xy - 3y^2 = 10$

11. Use the formal limit definition of the derivative to find the derivative of the following:

   (a) $f(x) = 3x^2 - x + 5$
   (b) $f(x) = \dfrac{2}{x-1}$
   (c) $f(x) = \sqrt{x+1}$

12. Use the quotient rule to derive the formula for the derivative of $\sec(x)$.

13. Given that $h(x) = f(g(x))$ and the following table of values:

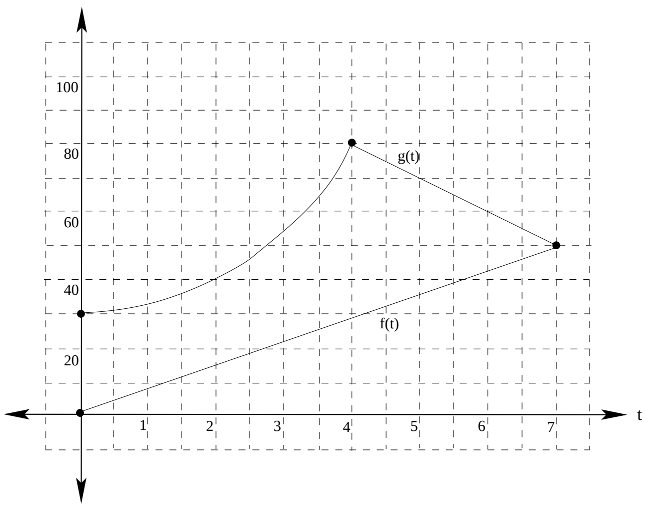
	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	7	-2	-1	2
4	3	1	0	2

Find the following:

(a) $(f+g)(1)$
(b) $(f+g)'(1)$
(c) $(fg)(4)$
(d) $(fg)'(4)$
(e) $\left(\frac{f}{g}\right)(1)$
(f) $h'(4)$

14. If $f(x) = \sqrt{3-2x}$, find the intervals where $f(x)$ is continuous, and find the intervals where $f(x)$ is differentiable.

15. The total net value of two companies: Company A and Company B, in millions of dollars as a function of time (in years since the year 2000) is given by $f(t)$ and $g(t)$ respectively.



(a) How fast was company A's value growing in 2005? (include units for your answer)
(b) What was the average rate of change for company B from 2000 through 2007?
(c) Which company's value was growing faster in 2004?
(d) Which company's value was growing faster in 2001?
(e) Which company would you rather own stock in, and why?

16. Find the equation for the tangent line to $f(x) = \sqrt[3]{x-5}$ when $x = 13$. Then, use this tangent line to approximate $\sqrt[3]{10}$.

17. Find the equation of the tangent line to the graph of $f(x) = \sin(2x)$ when $x = \frac{\pi}{6}$

18. Find the tangent line to the graph of the relation $4xy - 4x^2 = y^2$ at the point $(1, 2)$

19. Prove the Quotient Rule by applying the the Chain Rule to the general function $h(x) = f(x)[g(x)]^{-1}$.

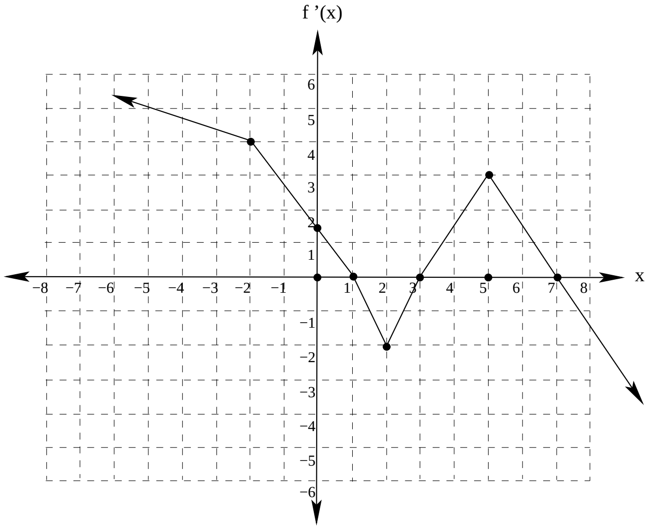
20. A person flying a kite holds the string 5 feet above ground level, and the string is played out at a rate of 2 feet per second as the kite moves horizontally at an altitude of 105 feet. Assuming that there is no sag in the string, find the rate at which the kite is moving when a total of 125 feet of string has been let out.

21. Draw the graph of a function $f(x)$ that is continuous when $x = 4$, but is not differentiable when $x = 4$.

22. Find $f^{(6)}(x)$ if $f(x) = \cos(2x)$

23. Find $f^{(21)}(x)$ if $f(x) = x^{20} + 7x^5 - 3x^3 - 1$

24. The graph of f' is given. Answer the following questions.



(a) Find the intervals where f is increasing.

(b) Find the intervals where f is decreasing.

(c) Find the location of all local maximums.

(d) Find the location of all local minimums.

(e) Find the intervals where f is concave up.

(f) Find the intervals where f is concave down.

(g) Find any inflection points.

(h) Sketch a possible graph for $f(x)$.

(i) Find the location of the absolute maximum on $[0, 7]$, if one exists.

(j) Find the location of the absolute minimum on $[0, 7]$, if one exists.

(k) Sketch a possible graph of f''

25. Find the absolute maximum and the absolute minimum of each of the following functions on the given interval.

(a) $f(x) = x^2 - 10x + 12$ on the interval $[-1, 7]$

(b) $f(x) = x^2 - 6x + 7$ on the interval $[-2, 2]$

(c) $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$ on the interval $[0, 4]$

(d) $f(x) = \frac{x+1}{2x-3}$ on the interval $[2, 5]$

(e) $f(x) = \frac{x}{2x-1}$ on the interval $[0, 4]$

26. Sketch a graph which satisfies the following:

- domain is $\{x|x \neq -3\}$
- y-intercept is $(0, -1)$
- x-intercepts are $(-4, 0)$, $(-1, 0)$, and $(7, 0)$
- the vertical asymptote is at $x = -3$
- $\lim_{x \to -\infty} f(x) = -2$
- $\lim_{x \to \infty} f(x) = 3$
- increasing on $(-\infty, -3) \cup (4, \infty)$
- decreasing on $(-3, 4)$
- local min at $(4, -5)$
- no local maxima
- concave up on $(-\infty, -3) \cup (-3, 0) \cup (2, 7)$
- concave down on $(0, 2) \cup (7, \infty)$
- inflection points at $(0, -1), (2, -2), (7, 0)$

27. A particle is moving along a straight line during a sixty-second time interval. Every ten seconds the position, velocity, and acceleration of the particle are measured, and recorded in the table below. For each of the following questions, answer the question and justify your answer.

t (seconds)	0	10	20	30	40	50	60
$s(t)$ (feet)	0	15	20	35	80	50	20
$v(t)$ (ft/s)	0	2	3	4	2	3	0
$a(t)$ (ft/s^2)	2	1	0	1	0	5	1

(a) Is there a time when $s(t) = 45$ feet?

(b) Is there a time when $v(t) = 1$ ft/sec?

(c) Is there a time when $v(t) = .5$ ft/sec?

(d) Is there a time when $v(t) = -3$ ft/sec?

(e) What is the largest velocity that you can justify from the data in the table?

(f) Is there a time when $a(t) = 3$ ft/sec^2?

(g) Is there a time when $a(t) = -0.2$ ft/sec^2?

28. Find a number c in the given interval that satisfies the Mean Value Theorem for the function and interval given, or explain why the Mean Value Theorem does not apply.

(a) $f(x) = x^4 + 2x$ on $[-1, 1]$

(b) $f(x) = \dfrac{x^2}{x - 1}$ on $[-2, 2]$

29. A hotel that charges \$80 per day for a room gives special rates to organizations that reserve between 30 and 60 rooms. If more than 30 rooms are reserved, the charge per room is decreased by \$1 times the number of rooms over 30. Under these terms, what number of rooms gives the maximum income?

30. A page of a book is to have an area of 90 square inches with 1 inch margins on the bottom and sides of the page and a 1/2-inch margin at the top. Find the dimensions of the page that would allow the largest printed area.

31. Sketch the following functions using information about the domain, the intercepts, the intervals where the function is increasing or decreasing, any local maximum or minimum, the intervals where it is concave up or concave down, any inflection points, and any asymptotes.

   (a) $f(x) = 2x^3 - 6x^2 - 18x$
   (b) $f(x) = \dfrac{x}{2x-1}$
   (c) $f(x) = \sin x + \cos x$

32. Suppose you are standing on the roof of a shed that is 20 feet tall and you throw a ball vertically upward. If you release the ball 4 feet above the top of the shed at an initial speed of 40 feet per second, how high will the ball travel and when will the ball hit the ground? (Assume gravity is $-32ft/sec^2$)

33. Use Newton's Method to approximate a real root of the function $f(x) = x^3 - 3x^2 + 2$ to 5 decimal places.

34. Use Newton's Method to approximate $\sqrt[3]{6}$ to 5 decimal places.

35. Find each of the following indefinite integrals:

   (a) $\int x^2 + C\,dx$
   (b) $\int x^2 + C\,dC$
   (c) $\int \dfrac{3x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + 3}{x^{\frac{1}{2}}}\,dx$
   (d) $\int x^2\sqrt{x^3 - 5}\,dx$
   (e) $\int 4\tan^3 x \sec^2 x\,dx$
   (f) $\int \dfrac{x}{\sqrt{2x-1}}\,dx$

36. Solve the following differential equations under the given initial conditions:

   (a) $f'(x) = \cos x + 2x; f(0) = 5$
   (b) $g''(x) = 4\cos(2x) - 6\sin(3x); g'(\pi) = 4; g(0) = 6$

37. Express the following in summation notation:

   (a) $2 + 5 + 8 + 11 + 14 + 17$
   (b) $\frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \frac{5}{11} + \frac{6}{13} + \frac{7}{15} + \frac{8}{17}$

38. Evaluate the following sums:

   (a) $\sum_{2}^{5} k(2k-1)$
   (b) $\sum_{4}^{15} k^3 - 2k^2$

39. Express the following sums in terms of n:

   (a) $\sum_{k=1}^{n} k^3 - 3k + 5$
   (b) $\sum_{3}^{n} k(3 - k^2)$

40. Consider $f(x) = 2x^2 - 3$ in the interval $[2, 5]$

(a) Find a summation formula that gives an estimate the definite integral of f on $[2, 5]$ using n equal width rectangles and using left hand endpoints to give the height of each rectangle. You do not have to evaluate the sum or find the exact area.

(b) Find the norm of the partition $P : 2 < 3 < 3.5 < 4 < 4.5 < 5$

(c) Find the approximation of the definite integral of f on $[2, 5]$ using the Riemann sum for the partition P given in part (b).

41. Assume f is continuous on $[-1, 4]$, $\int_{-1}^{1} f(x)\, dx = 5$, $\int_{1}^{4} f(x)\, dx = -1$, and $\int_{2}^{4} f(x)\, dx = 2$. Find:

(a) $\int_{1}^{-1} f(x)\, dx$

(b) $\int_{-1}^{4} f(x)\, dx$

(c) $\int_{1}^{2} f(x)\, dx$

(d) $\int_{-1}^{2} f(x)\, dx$

(e) Find the average value of f on $[-1, 1]$

42. Evaluate the following:

(a) $\int_{1}^{4} 3x^2 + \sqrt{x} + 2\, dx$

(b) $\int_{0}^{1} x^2(x^3 + 5)^2\, dx$

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^3(3x) \sin(3x)\, dx$

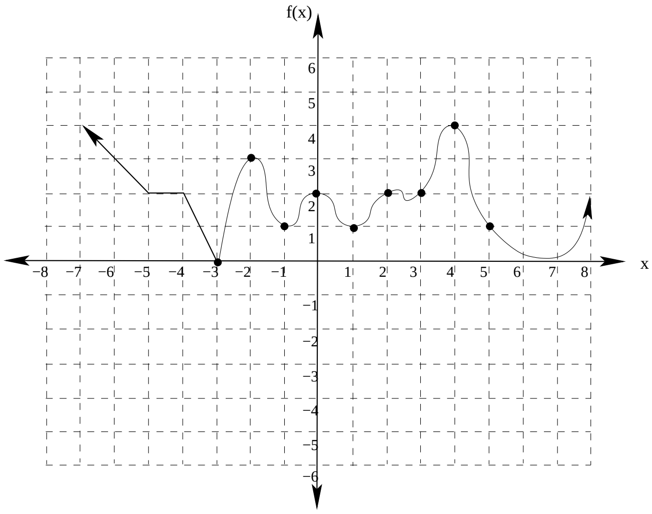
(d) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin x\, dx$

43. Compute the following:

(a) $\frac{d}{dx}\left(\int_{2}^{x^2-1} \frac{t}{\sqrt{t-1}}\, dt\right)$ (b) $\int \frac{d}{dt}\left(\frac{t}{\sqrt{t-1}}\right) dt$

(c) $\frac{d}{dx}\left(\int_{2}^{5} \frac{t}{\sqrt{t-1}}\, dt\right)$ (d) $\int_{2}^{5} \left[\frac{d}{dt}\left(\frac{t}{\sqrt{t-1}}\right) dt\right]$

44. Given the following graph of $f(x)$:



(a) Approximate $\int_{-3}^{5} f(x)\,dx$ using 8 equal width rectangles with height given by the right hand endpoint of each rectangle.

(b) Use the Trapezoid Rule with $n = 4$ to approximate $\int_{-3}^{5} f(x)\,dx$.

45. (a) Use the Trapezoidal Rule with $n = 4$ to approximate $\int_0^{\pi} 3\sin x\ dx$

(b) Find the maximum possible error in your approximation from part (a).

(c) Find the minimum number of rectangles that should be used to guarantee an approximation of $\int_0^{\pi} 3\sin x\ dx$ to within 4 decimal places using the Trapezoidal Rule.

(d) Use the Fundamental Theorem of Calculus to find $\int_0^4 2x^3\ dx$ exactly. How far off was your estimate? How does the actual error compare to the maximum possible error?

46. Graphs the following conic sections. Clearly label the main geometric features.

(a) $y^2 - 6y + 8x + 41$

(b) $x^2 + 4y^2 - 8x + 8y + 19 = 0$

(c) $9y^2 - 4x^2 - 54y + 45 = 0$

47. Find an equation for the conic section with the given features:

(a) A parabola with vertex $(-3, 5)$, axis parallel to the x-axis, and passing through the point $(5, 9)$

(b) An ellipse with center $(0, 0)$ passing through the points $(2, 3)$ and $(6, 1)$

(c) A hyperbola with foci $(0, \pm 3)$ and vertices $(0, \pm 2)$