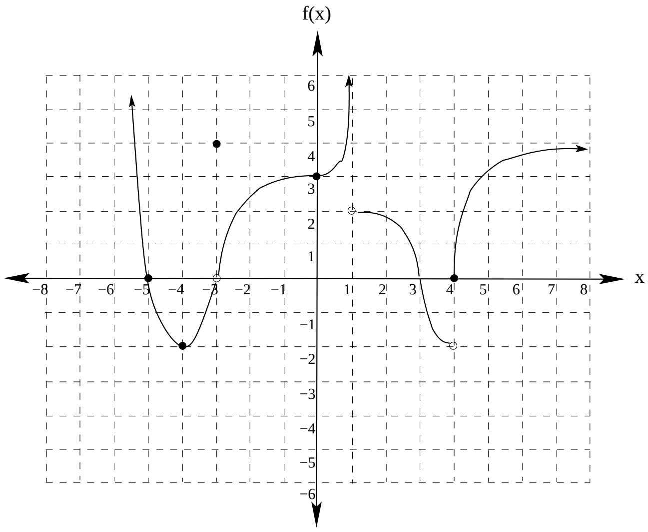
Math 261
Final Exam - Practice Problem Solutions

1. A function $f$ is graphed below.




(a) Find $f(0)$, $f(-2)$, $f(1)$, and $f(4)$
$f(0) = 3; f(-2) \approx 2.4; f(1)$ is undefined; $f(4) = 0$

(b) Find the domain and range of $f$
Domain: $(-\infty, 1) \cup (1, \infty)$ Range: $[-2, \infty)$

(c) Find the intervals where $f'(x)$ is positive
$f'(x) > 0$ on $(-4, 1) \cup (4, \infty)$

(d) Find the intervals where $f''(x)$ is negative.
$f''(x) < 0$ on $(-3, 0) \cup (1, 3) \cup (4, \infty)$

(e) Find $\lim_{x \to -2} f(x)$
$\lim_{x \to -2} f(x) \approx 2.4$

(f) find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$
$\lim_{x \to 4^-} f(x) = -2; \lim_{x \to 4^+} f(x) = 0$

(g) find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$
$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \approx 2.9$

(h) find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$
$\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 4$

(i) find the points where $f(x)$ is discontinuous, and classify each point of discontinuity.
$f$ has a removable discontinuity at $x = -3$, an infinite discontinuity at $x = 1$, and a jump discontinuity at $x = 4$.

2. Evaluate the following limits:

(a) $\lim_{x \to 1} \frac{2x^2 - 5x - 3}{3x^2 - 4x - 15} = \lim_{x \to 1} \frac{(2x+1)(x-3)}{(3x+5)(x-3)} = \frac{(2x+1)}{(3x+5)} = \frac{3}{8}$

(b) $\lim_{x \to 3} \frac{2x^2 - 5x - 3}{3x^2 - 4x - 15} = \lim_{x \to 3} \frac{(2x+1)(x-3)}{(3x+5)(x-3)} = \frac{(2x+1)}{(3x+5)} = \frac{7}{14} = \frac{1}{2}$

(c) $\lim_{x \to 2} \sqrt{2x - 4}$ is undefined since $\sqrt{2x - 4}$ is only defined for $x \geq 2$.

(d) $\lim_{x \to \pi} \cos x = -1$

(e) $\lim_{x\to\infty} \cos x$ is undefined ($\cos x$ continues to oscillate from 1 to $-1$ and back)

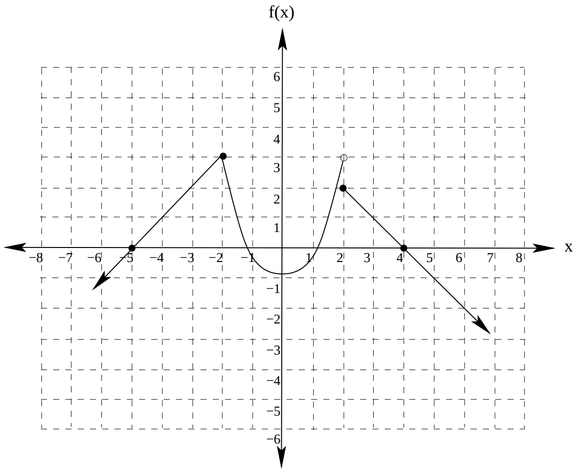
(f) $\lim_{x\to\infty} \frac{2x^2-5x-3}{3x^2-4x-15} = \frac{2}{3}$

(g) $\lim_{x\to\infty} \frac{2x^2-5x-3}{3x^3-4x-15} = 0$

3. Given the function

$$f(x) = \begin{cases} x+5 & \text{if } x \le -2 \\ x^2-1 & \text{if } |x| < 2 \\ 4-x & \text{if } x \ge 2 \end{cases}$$

(a) Graph $f(x)$.




(b) Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to -2} f(x)$

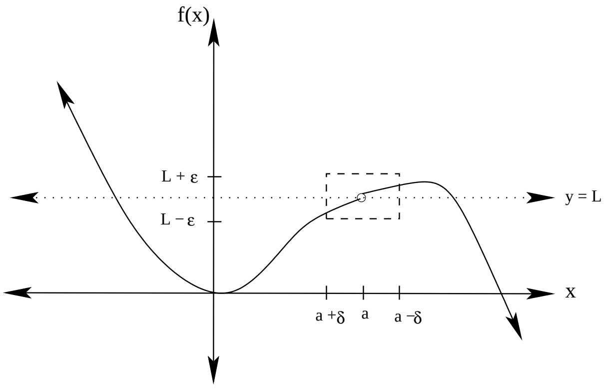
$\lim_{x\to 2^-} f(x) = 3$, $\lim_{x\to 2^+} f(x) = 2$, and $\lim_{x\to -2} f(x) = 3$

(c) Is $f(x)$ continuous at $x = 1$? Justify your answer.

Yes. In an interval containing $x = 1$, the function $f$ is defined by $x^2 - 1$ which is a polynomial and hence is continuous.

4. Give the formal $\epsilon$ - $\delta$ definition of the limit of a function as presented in class. Then draw a diagram illustrating the definition. Finally, write the definition informally in your own words.

Let $f$ be defined on an open interval containing $a$, except possibly at $a$ itself, and let $L$ be a real number. The statement $\lim_{x\to a} f(x) = L$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.




Intuitively, what the formal definition of a limit says is that $\lim_{x\to a} f(x) = L$ means that if we set an error tolerance of $\epsilon$ on the $y$-axis, then no matter how small we set our error tolerance, it is possible to choose an error tolerance $\delta$ on the $x$-axis so that all points within $\delta$ of $a$ on the $x$-axis get mapped by the function to points that are within $\epsilon$ of our limit value $L$.

5. Given that $f(x) = 3x^2 - 1$, $\lim_{x\to 1} f(x) = 2$, and $\epsilon = .01$, find the largest $\delta$ such that if $0 < |x-1| < \delta$, then $|f(x)-2| < \epsilon$.

   We need $|f(x) - 2| < \epsilon$. That is $|3x^2 - 1 - 2| < .01$ or $|3x^2 - 3| < .01$

   Therefore, $-.01 < 3x^2 - 3 < .01$, or $2.99 < 3x^2 < 3.01$, so $\frac{2.99}{3} < x^2 < \frac{3.01}{3}$.

   Hence $\sqrt{\frac{2.99}{3}} < x < \sqrt{\frac{3.01}{3}}$ or, rounding, $.998331942 < x < 1.001166528$.

   Thus $-.00166806 < x - 1 < .00166528$. So we can take $\delta < \sqrt{\frac{3.01}{3}} - 1 \approx .00166528$

6. Use the formal definition of a limit to prove that $\lim_{x\to 2} 5 - 2x = 1$.

   Suppose $|f(x) - L| < \epsilon$. Then $|5 - 2x - 1| < \epsilon$ or $|4 - 2x| < \epsilon$.

   That is, $|2(2 - x)| < \epsilon$, or $2|2 - x| < \epsilon$, so $|x - 2| < \frac{\epsilon}{2}$.

   Let $\delta \leq \frac{\epsilon}{2}$. Then if $0 < |x - 2| < \delta \leq \frac{\epsilon}{2}$, $2|x - 2| < \epsilon$.

   Therefore $|2x - 4| = |4 - 2x| = |5 - 2x - 1| < \epsilon$. That is, $|f(x) - 1| < \epsilon$.

   Hence $\lim_{x\to 2} f(x) = 1$

7. Let $f(x) = \dfrac{2x^2 - 4x}{x^2 - x - 2}$.

   (a) Find the values of $x$ at which $f$ is discontinuous.

   Notice that $f(x) = \dfrac{2x^2 - 4x}{x^2 - x - 2} = \dfrac{2x(x-2)}{(x-2)(x+1)}$.

   Since $f$ is a rational function, it is continuous everywhere except where the denominator is zero. Therefore, $f$ is discontinuous at $x = 2$ and at $x = -1$.

   (b) Find all vertical and horizontal asymptotes of $f$.

   Since the discontinuity at $x = 2$ is removable, $f$ only has one vertical asymptote at $x = -1$.

   Since $\lim_{x\to\infty} f(x) = 2$, $f$ has a horizontal asymptote at $y = 2$.

8. Find the $x$ values at which $f(x) = \sqrt{3 - 2x} + \dfrac{1}{\sqrt{2x + 5}}$ is continuous.

   Since $f$ consists of a sum of square roots of polynomial functions, $f$ is continuous wherever it is defined. Thus we need $3 - 2x \geq 0$, or $x \leq \frac{3}{2}$ and we need $2x + 5 > 0$, or $x > -\frac{5}{2}$. Thus $f$ is continuous on the interval $(-\frac{5}{2}, \frac{3}{2}]$.

9. (a) Use the Intermediate Value Theorem to show $f(x) = 2x^3 + 3x - 4$ has a root between 0 and 1.

   First notice that $f$ is a polynomial, and hence is continuous everywhere, so the IVT applies in this situation. Next, $f(0) = -4$, and $f(1) = 1$, so by the IVT, $f$ attains every value between -4 and 1 at least once for some $x$-value in the interval $(0, 1)$. In particular, there must be some $c$ in the interval $(0, 1)$ with $f(c) = 0$.

   (b) Use Newton's method to approximate this root to 4 decimal places.

   Recall that Newton's method uses the derivative to recursively approximate a root of a function. Given an initial guess $x_0$, we compute approximations using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

   Here, $f'(x) = 6x^2 + 3$, and we will take $x_0 = .5$.

   Then $x_1 = 1$; $x_2 = .8888889$, $x_3 = .8796739$, $x_4 = .87961488$, $x_5 = .87961488$, so our approximation of the root to four decimal places is :$x = .8796$.

10. Find the derivative $y' = \frac{dy}{dx}$ for each of the following:

    (a) $y = \pi^3 + \pi^2 x - \pi x^3 + x^\pi$

    $y' = \pi^2 - 3\pi x^2 + \pi x^{\pi - 1}$

    (b) $y = \cos(3x) + \sin(3x)$

    $y' = -3\sin(3x) + 3\cos(3x)$

    (c) $y = x^4 + \cos(x^4)$

    $y' = 4x^3 - 4x^3 \sin(x^4)$

    (d) $y = \sqrt{x} \tan x$

    $y' = \frac{1}{2}x^{-\frac{1}{2}} \tan x + x^{\frac{1}{2}} \sec^2(x)$

(e) $y = \sec^3(x^3)$
$y' = 3\sec^2(x^3) \cdot \sec(x^3)\tan(x^3) \cdot 3x^2 = 9x^2 \sec^3(x^3)\tan(x^3)$

(f) $y = \frac{3-x}{x^2+1}$
$y' = \frac{(1)(x^2+1)-(3-x)(2x)}{(x^2+1)^2} = \frac{x^2-6x-1}{(x^2+1)^2}$.

(g) $y = \dfrac{x^2 \cos x}{x + \sin(3-2x)}$
$y' = \frac{(2x\cos x - x^2 \sin x)(x+\sin(3-2x))-(x^2\cos x)(1-2\cos(3-2x))}{(x+\sin(3-2x))^2}$

(h) $y = \sin^2(\tan(x^3-5))$
$y' = 2\sin(\tan(x^3-5))\cos(\tan(x^3-5))\sec^2(x^3-5) \cdot 3x^2$

(i) $x^2 - 3xy + y^2 = 0$
Differentiating implicitly, $2x - 3y - 3xy' + 2yy' = 0$, so $2yy' - 3xy' = 3y - 2x$, or $y'(2y-3x) = 3y - 2x$.
Hence $y' = \frac{3y-2x}{2y-3x}$

(j) $2x^2y - 5xy - 3y^2 = 10$ Differentiating implicitly, $4xy + 2x^2y' - 5y - 5xy' - 6yy' = 0$, so $4xy - 5y = -2x^2y' + 5xy' + 6yy' = y'(-2x^2+5x+6y)$. Hence $y' = \frac{4xy-5y}{-2x^2+5x+6y}$

11. Use the formal limit definition of the derivative to find the derivative of the following:

(a) $f(x) = 3x^2 - x + 5$
$$f'(x) = \lim_{h\to 0} \frac{3(x+h)^2 - (x+h) + 5 - (3x^2 - x + 5)}{h} = \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x - 5}{h}$$
$$= \lim_{h\to 0} \frac{6xh + 3h^2 - h}{h} = \lim_{h\to 0} 6x + 3h - 1 = 6x - 1$$

(b) $f(x) = \dfrac{2}{x-1}$
$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{x+h-1} - \frac{2}{x-1}}{h} = \lim_{h\to 0} \frac{\frac{2(x-1)-2(x+h-1)}{(x+h-1)(x-1)}}{h}$$
$$= \lim_{h\to 0} \frac{2x - 2 - 2x - 2h + 2}{(x+h-1)(x-1)} \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-2h}{(x+h-1)(x-1)} \frac{1}{h} = \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} = \frac{-2}{(x-1)^2}$$

(c) $f(x) = \sqrt{x+1}$
$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$
$$= \lim_{h\to 0} \frac{x+h+1-(x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

12. Use the quotient rule to derive the formula for the derivative of $\sec(x)$.
Recall that $\sec x = \frac{1}{\cos x}$, so by the quotient rule:
$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \frac{0-(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$.

13. Given that $h(x) = f(g(x))$ and the following table of values:

| | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 7 | -2 | -1 | 2 |
| 4 | 3 | 1 | 0 | 2 |

Find the following:

(a) $(f+g)(1) = f(1) + g(1) = 7 + (-2) = 5$

(b) $(f+g)'(1) = f'(1) + g'(1) = -1 + 2 = 1$

(c) $(fg)(4) = f(4) \cdot g(4) = 3 \cdot 1 = 3$

(d) $(fg)'(4) = f'(4)g(4) + f(4)g'(4) = 0 + (3)(2) = 6$

(e) $\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} = -\frac{7}{2}$

(f) $h'(4) = f'(g(4))g'(4) = f'(1)g'(4) = (-1)(2) = -2$

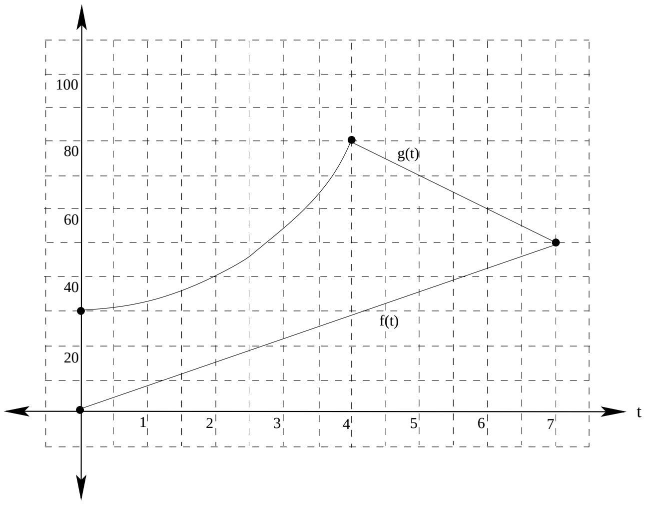
14. If $f(x) = \sqrt{3-2x}$, find the intervals where $f(x)$ is continuous, and find the intervals where $f(x)$ is differentiable.

First notice that $f$ is continuous wherever it is defined, so we need $3 - 2x \geq 0$, or $x \leq \frac{3}{2}$, so $f$ is continuous on $(-\infty, \frac{3}{2}]$.

Next, $f'(x) = \frac{1}{2}(3-2x)^{-\frac{1}{2}}(-2) = -(3-2x)^{\frac{1}{2}} = \frac{-1}{\sqrt{3-2x}}$

Therefore, $f$ is differentiable when $3 - 2x > 0$, or when $x < \frac{3}{2}$. That is, on the interval $(-\infty, \frac{3}{2})$.

15. The total net value of two companies: Company $A$ and Company $B$, in millions of dollars as a function of time (in years since the year 2000) is given by $f(t)$ and $g(t)$ respectively.




(a) How fast was company $A$'s value growing in 2005? (include units for your answer)
Notice that the growth of company $A$ in 2005 is given by the value of the derivative of $f$ when $t = 5$, but since $f$ is linear, its derivative is constant, so we have $f'(5) = \frac{50}{7} \approx 7.143$ in millions of dollars per year.

(b) What was the average rate of change for company $B$ from 2000 through 2007?
This is given by the slope of the secant line between $(0, g(0))$ and $(7, g(7))$.
Here $m_{av} = \frac{50-30}{7-0} = \frac{20}{7} \approx 2.857$ in millions of dollars per year.

(c) Which company's value was growing faster in 2004?
This is a bit open to interpretation since there is a cusp point on $g(t)$ when $t = 4$. Looking backwards, the value of company $B$ is increasing more steeply than that of company $A$, but the rate of change of company $B$ switches from positive to negative at $t = 4$.
Since we usually view time as moving forward, we will say that at the beginning of 2004, Company $B$'s value is growing faster.

(d) Which company's value was growing faster in 2001?
The situation here is much clearer. Looking at the tangent lines to each of the functions when $t = 1$, we see that the tangent line to $f(t)$ has larger slope, so Company $A$'s value is growing faster at the beginning of 2001.

(e) Which company would you rather own stock in, and why?
This question is a bit qualitative, since we cannot know what will happen in the future for thee companies, but it appears that Company $A$ is steadily increasing in value, while Company $B$ has topped out and is starting to decline. I would rather own stock in Company $A$.

16. Find the equation for the tangent line to $f(x) = \sqrt[3]{x-5}$ when $x = 13$. Then, use this tangent line to approximate $\sqrt[3]{10}$.

First notice that $f'(x) = \frac{1}{3}(x-5)^{-\frac{2}{3}}$, so the slope of the tangent line is $\frac{1}{3}(13-5)^{-\frac{2}{3}} = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.

Next, notice that $f(13) = \sqrt[3]{13-5} = \sqrt[3]{8} = 2$, so by point/slope, the tangent line is given by the equation $y - 2 = \frac{1}{12}(x - 13)$, or $y = \frac{1}{12}x + \frac{11}{12}$.

Now, $f(15) = \sqrt[3]{15-5} = \sqrt[3]{10}$. Then, using our tangent line to approximate:

$\sqrt[3]{10} \approx \frac{1}{12}(15) + \frac{11}{12} = \frac{15}{12} + \frac{11}{12} = \frac{26}{12} \approx 2.1666667.$

(In fact, calculating directly, $\sqrt[3]{10} \approx 2.15443469$)

17. Find the equation of the tangent line to the graph of $f(x) = \sin(2x)$ when $x = \frac{\pi}{6}$

    $f'(x) = 2\cos(2x)$, so $m = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$

    Also, $f(\frac{\pi}{6}) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

    Therefore, the tangent line equation is given by $y - \frac{\sqrt{3}}{2} = x - \frac{\pi}{6}$, or $y = x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

18. Find the tangent line to the graph of the relation $4xy - 4x^2 = y^2$ at the point $(1, 2)$

    Differentiating implicitly, $4y + 4xy' - 8x = 2yy'$, so $4y - 8x = 2yy' - 4xy' = y'(2y - 4x)$.

    Thus $y' = \frac{4y-8x}{2y-4x}$, which when $x = 1$ and $y = 2$ gives:$y' = \frac{4(2)-8}{2(2)-4} = \frac{0}{0}$ – uh oh!

    The tangent line is undefined! There is no tangent line at this point.

19. Prove the Quotient Rule by applying the the Chain Rule to the general function $h(x) = f(x)[g(x)]^{-1}$.

    If $h(x) = f(x)[g(x)]^{-1}$, then $h'(x) = f'(x)[g(x)]^{-1} + f(x) \cdot (-1)[g(x)]^{-2} \cdot g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f'(x)g(x)}{g(x)^2} - \frac{f(x)g'(x)}{g(x)^2}$
    $= \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

20. A person flying a kite holds the string 5 feet above ground level, and the string is played out at a rate of 2 feet per second as the kite moves horizontally at an altitude of 105 feet. Assuming that there is no sag in the string, find the rate at which the kite is moving when a total of 125 feet of string has been let out.

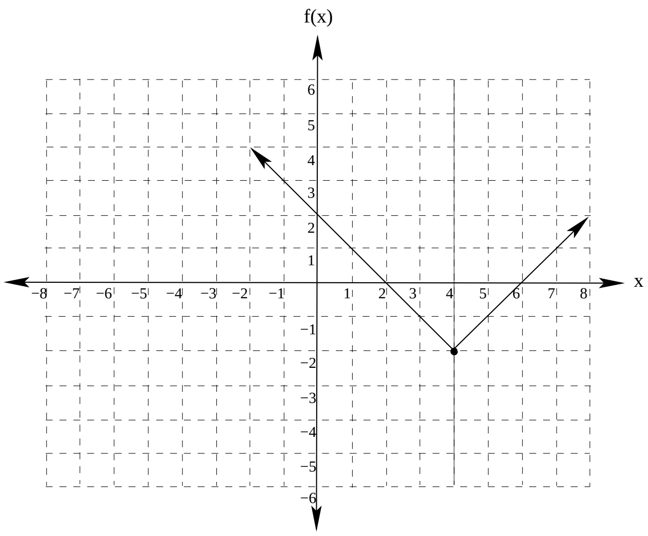
    Let $L$ be the length of the string. Let $x$ be the horizontal distance of the kite from the place where the person is standing. Then, since the net height of the kite is 100 feet (105 - 5 feet), using the Pythagorean Theorem, we have $L^2 = x^2 + 100^2$. [Note: since the motion is horizontal, $y$ is constant]

    Differentiating this implicitly, $2LL' = 2xx'$, or solving for $x'$, $x' = \frac{2LL'}{2x} = \frac{LL'}{x}$.

    Notice that $L = 125$, $L' = 2$, and $x = \sqrt{(125)^2 - 100^2} = 75$ at the time in question.

    Therefore, $x' = \frac{(125)(2)}{75} = \frac{10}{3} \frac{ft}{sec}$.

21. Draw the graph of a function $f(x)$ that is continuous when $x = 4$, but is not differentiable when $x = 4$.




22. Find $f^{(6)}(x)$ if $f(x) = \cos(2x)$

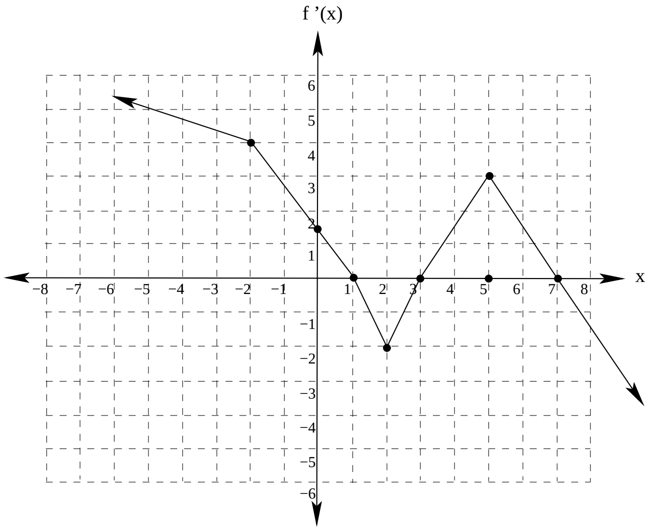
    $f'(x) = -2\sin(2x)$; $f''(x) = -4\cos(2x)$; $f^{(3)}(x) = 8\sin(2x)$;

    $f^{(4)}(x) = 16\cos(2x)$; $f^{(5)}(x) = -32\sin(2x)$; $f^{(6)}(x) = -64\cos(2x) = -2^6\cos(2x)$

23. Find $f^{(21)}(x)$ if $f(x) = x^{20} + 7x^5 - 3x^3 - 1$

    $f^{(21)}(x) = 0$. Each time you differentiate, the power drops by 1, so the 20th derivative is a constant. Hence, the 21st derivative is zero.

24. The graph of $f'$ is given. Answer the following questions.




(a) Find the intervals where $f$ is increasing.
$f$ is increasing whenever $f'$ is positive. That is, on the intervals: $(-\infty, 1) \cup (3, 7)$.

(b) Find the intervals where $f$ is decreasing.
$f$ is increasing whenever $f'$ is negative. That is, on the intervals: $(1, 3) \cup (7, \infty)$.

(c) Find the location of all local maximums.
There are local maxima whenever $f'$ goes from positive to negative, so at $x = 1$ and $x = 7$

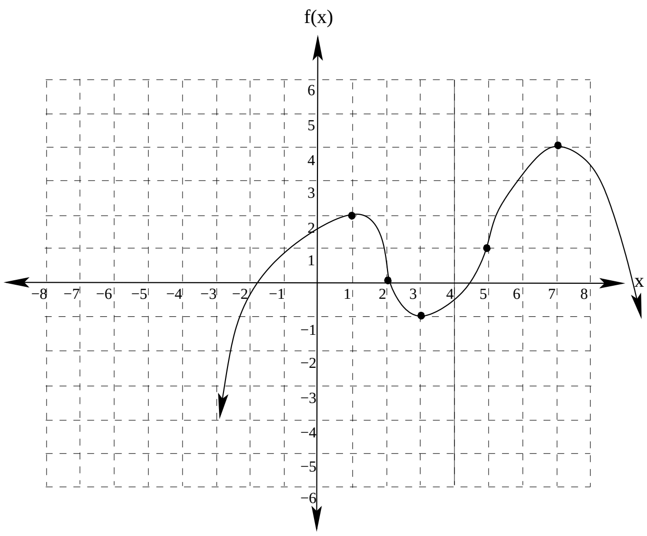
(d) Find the location of all local minimums.
There are local minima whenever $f'$ goes from negative to positive, so at $x = 3$

(e) Find the intervals where $f$ is concave up.
$f$ is concave up whenever $f'$ is increasing. That is, on the interval: $(2, 5)$

(f) Find the intervals where $f$ is concave down.
$f$ is concave down whenever $f'$ is decreasing. That is, on the intervals: $(-\infty, 2) \cup (5, \infty)$.

(g) Find any inflection points.
Inflection points occur when the $f$ changes concavity, so at: $x = 2$ and at $x = 5$

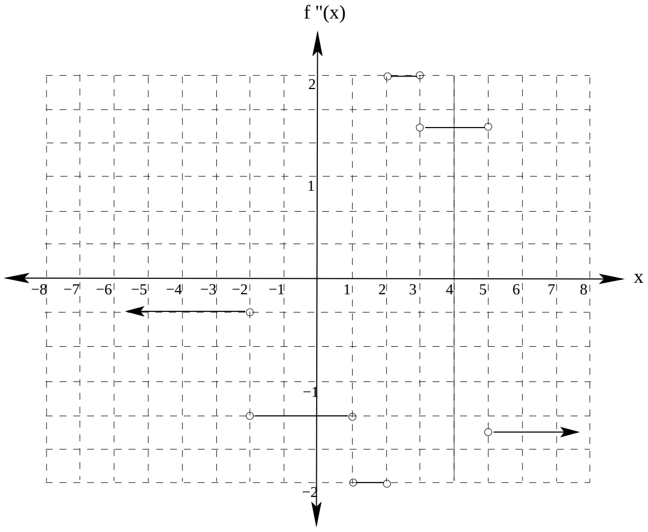
(h) Sketch a possible graph for $f(x)$.




(i) Find the location of the absolute maximum on $[0, 7]$, if one exists.
The absolute max is when $x = 7$. We see this by looking at areas under $f'(x)$ and convincing ourselves that $f(7)$ must be greater than $f(0)$ and $f(1)$.

(j) Find the location of the absolute minimum on $[0, 7]$, if one exists.
The absolute min is when $x = 3$ (this requires us to notice that the amount added to $f(x)$ as $x$ goes from 0 to 1 is less than the amount subtracted from $f(x)$ as $x$ goes from 1 to 3, so $f(0) > f(3)$).

(k) Sketch a possible graph of $f''$




25. Find the absolute maximum and the absolute minimum of each of the following functions on the given interval.

(a) $f(x) = x^2 - 10x + 12$ on the interval $[-1, 7]$
First notice that $f$ is continuous so the EVT applies. Next, $f'(x) = 2x - 10 = 0$ when $x = 5$
Now $f(-1) = 23$; $f(5) = -13$, and $f(7) = -9$, so the absolute max is: 23 and the absolute min is: -13

(b) $f(x) = x^2 - 6x + 7$ on the interval $[-2, 2]$
First notice that $f$ is continuous so the EVT applies. Next, $f'(x) = 2x - 6 = 0$ when $x = 3$, but this is not in our interval of interest.
Now $f(-2) = 23$ and $f(2) = -1$, so the absolute max is: 23 and the absolute min is: -1

(c) $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$ on the interval $[0, 4]$
First notice that $f$ is continuous so the EVT applies. Next, $f'(x) = x^2 - 4x + 3 = 0$ when $x = 1$ or $x = 3$.
Now $f(0) = 5$; $f(1) = \frac{19}{3}$, $f(3) = 5$ and $f(4) = \frac{19}{3}$, so the absolute max is: $\frac{19}{3}$ and the absolute min is: 5.

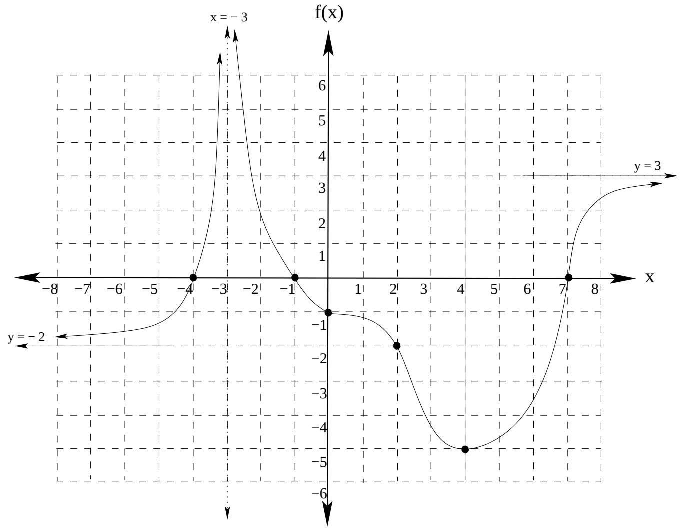
(d) $f(x) = \dfrac{x + 1}{2x - 3}$ on the interval $[2, 5]$
First notice that $f$ is continuous on $[2, 5]$ (the only value we need to worry about is $x = \frac{3}{2}$) so the EVT applies. Next, $f'(x) = \frac{(2x-3)-(x+1)2}{(2x-3)^2} = \frac{-5}{(2x-3)^2}$ which is never zero.
Now $f(2) = 3$, and $f(5) = \frac{6}{7}$, so the absolute max is: 3 and the absolute min is: $\frac{6}{7}$.

(e) $f(x) = \dfrac{x}{2x - 1}$ on the interval $[0, 4]$
Notice that $f$ is discontinuous at $x = \frac{1}{2}$, so the EVT does not apply. In fact, the function has a vertical asymptote that attains values approaching $\infty$ on one side and $-\infty$ on the other side, the this function has no absolute max and no absolute min on this interval.

26. Sketch a graph which satisfies the following:

- domain is $\{x | x \neq -3\}$
- $y$-intercept is $(0, -1)$
- $x$-intercepts are $(-4, 0)$, $(-1, 0)$, and $(7, 0)$
- the vertical asymptote is at $x = -3$
- $\lim_{x \to -\infty} f(x) = -2$
- $\lim_{x \to \infty} f(x) = 3$

- increasing on $(-\infty, -3) \cup (4, \infty)$
- decreasing on $(-3, 4)$ (oops - typo on original handout!)
- local min at $(4, -5)$
- no local maxima
- concave up on $(-\infty, -3) \cup (-3, 0) \cup (2, 7)$
- concave down on $(0, 2) \cup (7, \infty)$
- inflection points at $(0, -1), (2, -2), (7, 0)$




27. A particle is moving along a straight line during a sixty-second time interval. Every ten seconds the position, velocity, and acceleration of the particle are measured, and recorded in the table below. For each of the following questions, answer the question and justify your answer.

| $t$ (seconds) | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $s(t)$ (feet) | 0 | 15 | 20 | 35 | 80 | 50 | 20 |
| $v(t)$ (ft/s) | 0 | 2 | 3 | 4 | 2 | 3 | 0 |
| $a(t)$ (ft/s$^2$) | 2 | 1 | 0 | 1 | 0 | 5 | 1 |

(a) Is there a time when $s(t) = 45$ feet?
Yes. We may apply the IVT to the function $s(t)$ over the interval $[30, 40]$. Since $s(30) = 35 < 45 < s(40) = 80$ we may conclude that there is a $c$ in the interval $(30, 40)$ with $s(c) = 45$ feet.

(b) Is there a time when $v(t) = 1$ ft/sec?
Yes. We may apply the IVT to the function $v(t)$ over the interval $[0, 10]$. Since $v(0) = 0 < 1 < v(10) = 2$ we may conclude that there is a $c$ in the interval $(0, 10)$ with $v(c) = 1$ ft/sec.

(c) Is there a time when $v(t) = .5$ ft/sec?
Yes. We may apply the IVT to the function $v(t)$ over the interval $[0, 10]$. Since $v(0) = 0 < .5 < v(10) = 2$ we may conclude that there is a $c$ in the interval $(0, 10)$ with $v(c) = .5$ ft/sec.

(d) Is there a time when $v(t) = -3$ ft/sec?
Since $v(t) \geq 0$ for all values in the table, the IVT will not help us here. However, we may apply the MVT to the function $s(t)$ over the interval $[40, 50]$. Since $\dfrac{s(50) - s(40)}{50 - 40} = \dfrac{50 - 80}{10} = -3$ we may conclude that there is a $c$ in the interval $(40, 50)$ with $s'(c) = v(c) = -3$ ft/sec.

(e) What is the largest velocity that you can justify from the data in the table?
The largest value we can conclude using the IVT is 80 ft/sec. However, we may apply the MVT to the function $s(t)$ over the interval $[30, 40]$. Since $\dfrac{s(40) - s(30)}{40 - 30} = \dfrac{80 - 35}{10} = 4.5$ we may conclude that there is a $c$ in the interval $(30, 40)$ with $s'(c) = v(c) = 4.5$ ft/sec.

(f) Is there a time when $a(t) = 3$ ft/sec$^2$?
Yes. We may apply the IVT to the function $a(t)$ over the interval $[40, 50]$. Since $a(40) = 0 < 3 < a(50) = 5$ we may conclude that there is a $c$ in the interval $(0, 10)$ with $v(c) = .5$ ft/sec$^2$.

(g) Is there a time when $a(t) = -0.2$ ft/sec$^2$?

Since $a(t) \geq 0$ for all values in the table, the IVT will not help us here. However, we may apply the MVT to the function $v(t)$ over the interval $[30, 40]$. Since $\dfrac{v(40) - v(30)}{40 - 30} = \dfrac{2-4}{10} = -.2$ we may conclude that there is a $c$ in the interval $(30, 40)$ with $v'(c) = a(c) = -.2$ ft/sec$^2$.

28. Find a number $c$ in the given interval that satisfies the Mean Value Theorem for the function and interval given, or explain why the Mean Value Theorem does not apply.

(a) $f(x) = x^4 + 2x$ on $[-1, 1]$

First, notice that $f$ is continuous on $[-1, 1]$ so the MVT applies. Also, $\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{3 - (-1)}{2} = 2$. Next, $f'(x) = 4x^3 + 2$, so we need to find $x$ so that $4x^3 + 2 = 2$. That is, $4x^3 = 0$, or $x^3 = 0$. Thus $x = 0$ is the value that satisfies the Mean Value Theorem on the interval $[-1, 1]$.

(b) $f(x) = \dfrac{x^2}{x-1}$ on $[-2, 2]$

Here, notice that $f(x)$ has a vertical asymptote when $x = 1$, thus the MVT does not apply.

29. A hotel that charges \$80 per day for a room gives special rates to organizations that reserve between 30 and 60 rooms. If more than 30 rooms are reserved, the charge per room is decreased by \$1 times the number of rooms over 30. Under these terms, what number of rooms gives the maximum income?

Let $x$ be the number of rooms reserved. We know $0 \leq x \leq 60$, but since there is no discount for below 30 rooms, we know that $x \geq 30$.

The price per room for between 30 and 60 rooms is given by: $p(x) = 80 - (x - 30) = 110 - x$. Therefore, the revenue from $x$ rooms is given by $R(x) = x(110 - x) = 110x - x^2$.

Now $R'(x) = 110 - 2x = 0$ when $110 = 2x$, or when $x = 55$.

Thus, applying the EVT, since $R(30) = 2400$, $R(55) = 3025$ and $R(60) = 3000$, so the maximum revenue occurs when 55 rooms are reserved.

30. A page of a book is to have an area of 90 square inches with 1 inch margins on the bottom and sides of the page and a 1/2-inch margin at the top. Find the dimensions of the page that would allow the largest printed area.

Let $x$ be the width of the page and $y$ the length (in inches). Then $xy = 90$, or $y = \frac{90}{x}$. Accounting for the margins, we can see that the area of part of the page used for printing is given by: $A = (x-2)(y-\frac{3}{2}) = (x-2)(\frac{90}{x} - \frac{3}{2}) = 93 - \frac{180}{x} - \frac{3}{2}x^2$.

Therefore, $A'(x) = \frac{180}{x^2} - \frac{3}{2} = 0$ when $\frac{180}{x^2} = \frac{3}{2}$, or $x^2 = \frac{360}{3} = 120$, so $x = \sqrt{120}$. Notice that $A''(x) = -\frac{360}{x^3} < 0$ for $x > 0$, so this is a maximum.

Therefore the page should have dimensions $x = \sqrt{120} \approx 10.95$ inches and $y = \frac{90}{\sqrt{120}} \approx 8.22$ inches.

31. Sketch the following functions using information about the domain, the intercepts, the intervals where the function is increasing or decreasing, any local maximum or minimum, the intervals where it is concave up or concave down, any inflection points, and any asymptotes.

(a) $f(x) = 2x^3 - 6x^2 - 18x$

(b) $f(x) = \dfrac{x}{2x-1}$

(c) $f(x) = \sin x + \cos x$

[I'll add these in later if I have time. Sorry!]

32. Suppose you are standing on the roof of a shed that is 20 feet tall and you throw a ball vertically upward. If you release the ball 4 feet above the top of the shed at an initial speed of 40 feet per second, how high will the ball travel and when will the ball hit the ground? (Assume gravity is $-32ft/sec^2$)

From the description above, we see that $s(0) = 24$, $v(0) = 40$, and $a(t) = -32$.

Then $v(t) = -32t + 40$, and $s(t) = -16t^2 + 40t + 24$.

The ball will attain its maximum height when $v(t) = 0$, that is when $32t = 40$, or when $t = \frac{5}{4}$. Then the maximum height is $s(\frac{5}{4}) = -16(\frac{5}{4})^2 + 40(\frac{5}{4}) + 24 = -25 + 50 + 24 = 49$ feet.

The ball will hit the ground when $s(t) = 0$, that is, when $-16t^2 + 40t + 24 = -8(2t^2 - 5t - 3) = -8(2t + 1)(t - 3) = 0$, which has solutions $t = -\frac{1}{2}$ and $t = 3$. Clearly the negative solution does not make sense, so the ball must hit the ground after 3 seconds.

33. Use Newton's Method to approximate a real root of the function $f(x) = x^3 - 3x^2 + 2$ to 5 decimal places.

A clever student will notice that $f(1) = 0$, but that is not really using Newton's method, so we will pretend that we didn't notice this and apply Newton's method.

Recall that Newton's method uses the derivative to recursively approximate a root of a function. Given an initial guess $x_0$, we compute approximations using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Here, $f'(x) = 3x^2 - 6x$, notice that $f'(x) = 0$ when $x = 0$ and when $x - 2$ so we need to avoid these values (why?). We will take $x_0 = 1.5$.

Then $x_1 = .888888889$; $x_2 = 1.000925926$, $x_3 = .999999999$, $x_4 = 1$ (we are probably rounding too soon here, but that's all the precision my calculator has), $x_5 = 1$, ..., so our approximation of the root to five decimal places is :$x = 1.00000$.

34. Use Newton's Method to approximate $\sqrt[3]{6}$ to 5 decimal places.

The first trick here is to come up with a polynomial function that has $\sqrt[3]{6}$ as a root (well, I suppose it wouldn't have to be a polynomial function, but they are pretty easy to come by). If we start with $x = \sqrt[3]{6}$, then, cubing each side, we have $x^3 = 6$, so we may take $f(x) = x^3 - 6$.

Then $f'(x) = 3x^2$.

Newton's method uses the derivative to recursively approximate a root of a function. Given an initial guess $x_0$, we compute approximations using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Since we know that $1 < \sqrt[3]{6} < 2$, then we will take $x_0 = 2$.

Then $x_1 = 1.833333333$; $x_2 = 1.817263545$, $x_3 = 1.817120604$, $x_4 = 1.817120593$, $x_5 = 1.817120593$, so our approximation of the root to five decimal places is :$x = 1.81712$.

35. Find each of the following indefinite integrals:

(a) $\displaystyle\int x^2 + C\,dx = \frac{1}{3}x^3 + Cx + D$

(b) $\displaystyle\int x^2 + C\,dC = x^2C + \frac{1}{2}C^2 + D$

(c) $\displaystyle\int \frac{3x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + 3}{x^{\frac{1}{2}}}\,dx = \int 3x - 5 + 3x^{-\frac{1}{2}} = \frac{3}{2}x^2 - 5x + 6x^{\frac{1}{2}} + C$

(d) $\displaystyle\int x^2\sqrt{x^3 - 5}\,dx$

Let $u = x^3 - 5$. Then $du = 3x^2dx$, so $\frac{1}{3}du = x^2dx$.

This gives the integral $\displaystyle\frac{1}{3}\int u^{\frac{1}{2}}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{9}u^{\frac{3}{2}} + C = \frac{2}{9}(x^3 - 5)^{\frac{3}{2}} + C$.

(e) $\displaystyle\int 4\tan^3 x \sec^2 x\,dx$

Let $u = \tan x$. Then $du = \sec^2 x dx$. This gives the integral $\displaystyle 4\int u^3\,du = u^4 + C = \tan^4 x + C$.

(f) $\displaystyle\int \frac{x}{\sqrt{2x - 1}}\,dx$

Let $u = 2x - 1$. Then $du = 2dx$ or $\frac{1}{2}du = dx$.

Solving $u = 2x - 1$ for $x$, we get $\frac{u+1}{2} = \frac{u}{2} + \frac{1}{2} = x$.

This allows us to rewrite the integral as $\displaystyle\frac{1}{2}\int \frac{\frac{u}{2} + \frac{1}{2}}{u^{\frac{1}{2}}}\,du = \int \frac{1}{4}u^{\frac{1}{2}} + \frac{1}{4}u^{-\frac{1}{2}}\,du = \frac{1}{6}u^{\frac{3}{2}} + \frac{1}{2}u^{\frac{1}{2}} + C$.

36. Solve the following differential equations under the given initial conditions:

(a) $f'(x) = \cos x + 2x; f(0) = 5$

Antidifferentiating, we have $f(x) = \sin x + x^2 + C$, so $f(0) = 5 = \sin 0 + 0^2 + C = C$, so $C = 5$.

Thus $f(x) = \sin x + x^2 + 5$

(b) $g''(x) = 4\cos(2x) - 6\sin(3x); g'(\pi) = 4; g(0) = 6$

Antidifferentiating, we have $g'(x) = 2\sin(2x) + 2\cos(3x) + C$, so $g'(\pi)4 = 2\sin(\pi) + 2\cos(\pi) + C = 0 - 2 + C$.

Therefore, $C = 6$, and $g'(x) = 2\sin(2x) + 2\cos(3x) + 6$

Next, $g(x) = -\cos(2x) + \frac{2}{3}\sin(3x) + 6x + D$, so $g(0) = 6 = -\cos(0) + \frac{2}{3}\sin(0) + 6(0) + D = -1 + D$. Thus $D = 7$

Hence $g(x) = -\cos(2x) + \frac{2}{3}\sin(3x) + 6x + 7$

37. Express the following in summation notation:

(a) $2+5+8+11+14+17=\sum_{k=1}^{6} 3k-1$

(b) $\frac{2}{5}+\frac{3}{7}+\frac{4}{9}+\frac{5}{11}+\frac{6}{13}+\frac{7}{15}+\frac{8}{17}=\sum_{k=1}^{7}\frac{k+1}{2k+3}$

38. Evaluate the following sums:

(a) $\sum_{k=2}^{5} k(2k-1)$
$=(8-2)+(18-3)+(32-4)+(50-5)=6+15+28+45=94.$
(or you can do these using summation formulas, you decide which is easier)

(b) $\sum_{k=4}^{15} k^3-2k^2$
Here, it is probably better to use summation formulas, although you could certainly just calculate and then add up the terms.
Recall that $\sum_{k=1}^{n} k^3=\left(\frac{(n)(n+1)}{2}\right)^2$ and $\sum_{k=1}^{n} k^2=\left(\frac{n(n+1)(2n+1)}{6}\right)$.
Then $\sum_{k=4}^{15} k^3-2k^2=\sum_{k=1}^{15} k^3-2k^2-\sum_{k=1}^{3} k^3-2k^2=\left[\left(\frac{(15)(16)}{2}\right)^2-2\left(\frac{15(16)(31)}{6}\right)\right]-\left[\left(\frac{3(4)}{2}\right)^2-2\left(\frac{3(4)(7)}{6}\right)\right]$
$=\left[(120)^2-(2480)\right]-\left[6^2-28\right]=11,912$

39. Express the following sums in terms of $n$:

(a) $\sum_{k=1}^{n} k^3-3k+5$
$=\left(\frac{n(n+1)}{2}\right)^2-3\left(\frac{n(n+1)}{2}\right)+5n=\frac{n^4+2n^3+n^2}{4}-\frac{-3n^2-3n}{2}+5n=\frac{n^2}{4}+\frac{n^3}{2}-\frac{5n^2}{4}+\frac{7n}{2}$

(b) $\sum_{3}^{n} k(3-k^2)=\sum_{3}^{n} 3k-k^3=\sum_{1}^{n} 3k-k^3-(3-1)-(6-8)$
$=3\left(\frac{n(n+1)}{2}\right)-\left(\frac{n(n+1)}{2}\right)^2-2+2=\left(\frac{3n^2}{2}+\frac{3n}{2}\right)-\left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}\right)=-\frac{n^4}{4}-\frac{n^3}{2}+\frac{5n^2}{4}+\frac{3n}{2}$

40. Consider $f(x)=2x^2-3$ in the interval $[2,5]$

(a) Find a summation formula that gives an estimate the definite integral of $f$ on $[2,5]$ using $n$ equal width rectangles and using left hand endpoints to give the height of each rectangle. You do not have to evaluate the sum or find the exact area.
First notice that $\Delta x=\frac{5-2}{n}=\frac{3}{n}$.
Then $A=\sum_{k=1}^{n} f(2+(k-1)\Delta x)\Delta x=\sum_{k=1}^{n}[2(2+(k-1)\frac{3}{n})^2-3]\frac{3}{n}$
$=\sum_{k=1}^{n}\left[2\left(4+\frac{4n(k-1)}{3}+\frac{n^2(k-1)^2}{9}\right)-3\right]\frac{3}{n}=\sum_{k=1}^{n}\left[8+\frac{8n(k-1)}{3}+\frac{2n^2k^2-4n^2k+2n^2}{9}-3\right]\frac{3}{n}$

(b) Find the norm of the partition $P: 2<3<3.5<4<4.5<5$
The norm is the width of the largest subinterval, or 1.

(c) Find the approximation of the definite integral of $f$ on $[2,5]$ using the Riemann sum for the partition $P$ given in part (b).
Since we are using rectangles with left hand endpoints, the sum is given by:
$A\approx(1)(f(2))+(.5)(f(3))+(.5)(f(3.5))+(.5)(f(4))+(.5)(f(4.5))=(5)+(7.5)+(10.75)+(14.5)+(18.75)=56.5$

41. Assume $f$ is continuous on $[-1,4]$, $\int_{-1}^{1} f(x)\ dx = 5$, $\int_{1}^{4} f(x)\ dx = -1$, and $\int_{2}^{4} f(x)\ dx = 2$. Find:

(a) $\int_{1}^{-1} f(x)\ dx = -5$

(b) $\int_{-1}^{4} f(x)\ dx = 5 - 1 = 4$

(c) $\int_{1}^{2} f(x)\ dx = -1 - 2 = -3$

(d) $\int_{-1}^{2} f(x)\ dx = 5 - 3 = 2$

(e) Find the average value of $f$ on $[-1,1]$

The average of $f$ on $[-1,1]$ is given by: $\frac{1}{1-(-1)} \int_{-1}^{1} f(x)\,dx = \frac{1}{2}5 = \frac{5}{2}$.

42. Evaluate the following:

(a) $\int_{1}^{4} 3x^2 + \sqrt{x} + 2\ dx$

$= x^3 + \frac{2}{3}x^{\frac{3}{2}} + 2x\Big|_1^4 = (64 + \frac{16}{3} + 8) - (1 + \frac{2}{3} + 2) = \frac{221}{3} \approx 73.667$

(b) $\int_{0}^{1} x^2(x^3+5)^2\ dx$

Let $u = x^3 + 5$ Then $du = 3x^2\,dx$, or $\frac{1}{3}du = x^2\,dx$. Also, $0^3 + 5 = 5$ and $1^3 + 5 = 6$

This gives the integral: $\frac{1}{3}\int_{5}^{6} u^2\ du = \frac{1}{3}u^3\Big|_5^6 = \frac{216}{9} - \frac{125}{9} = \frac{91}{9} \approx 10.111$

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^3(3x)\sin(3x)\ dx$

Let $u = \cos(3x)$ Then $du = -3\sin(3x)\,dx$, or $-\frac{1}{3}du = \sin(3x)\,dx$. Also, $\cos(\frac{3\pi}{6}) = 0$ and $\cos(\frac{3\pi}{2}) = 0$.

This gives the integral: $-\frac{1}{3}\int_{0}^{0} u^3\ du = 0$

(d) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin x\ dx$

Notice that $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin x\ dx = 0$ since $\sin x$ is odd.

43. Compute the following:

(a) $\frac{d}{dx}\left(\int_{2}^{x^2-1} \frac{t}{\sqrt{t-1}}\ dt\right)$ (b) $\int \frac{d}{dt}\left(\frac{t}{\sqrt{t-1}}\right)\ dt$

(a) Using the first part of the fundamental theorem of calculus, $\frac{d}{dx}\left(\int_{2}^{x^2-1} \frac{t}{\sqrt{t-1}}\ dt\right) = \frac{d}{dx}\big(F(x^2-1) - F(2)\big)$

$= \frac{x^2-1}{\sqrt{x^2-1-1}} \cdot 2x = \frac{2x^3-2x}{\sqrt{x^2-2}}$

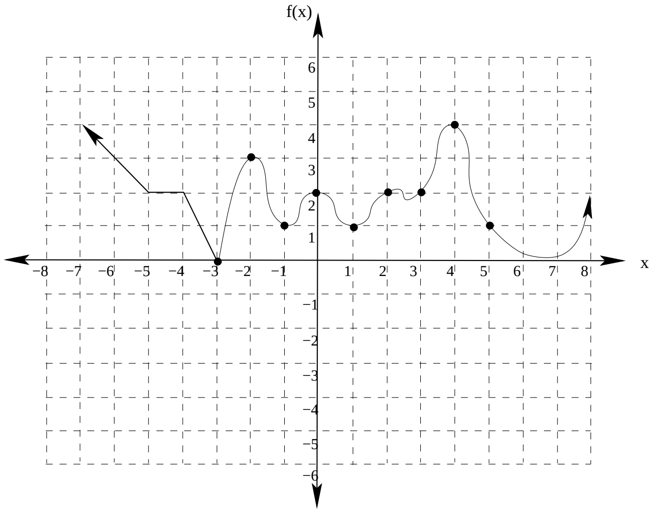
(b) Again applying the the fundamental theorem of calculus, $\int \frac{d}{dt}\left(\frac{t}{\sqrt{t-1}}\right)\ dt = \frac{t}{\sqrt{t-1}}$

(c) $\frac{d}{dx}\left(\int_{2}^{5} \frac{t}{\sqrt{t-1}}\ dt\right)$ (d) $\int_{2}^{5}\left[\frac{d}{dt}\left(\frac{t}{\sqrt{t-1}}\right)\ dt\right]$

(c) Since $\int_{2}^{5} \frac{t}{\sqrt{t-1}}\ dt$ is a constant, $\frac{d}{dx}\left(\int_{2}^{5} \frac{t}{\sqrt{t-1}}\ dt\right) = 0$

(d) $\int_{2}^{5}\left[\frac{d}{dt}\left(\frac{t}{\sqrt{t-1}}\right)\ dt\right] = \frac{5}{\sqrt{5-1}} - \frac{2}{\sqrt{2-1}} = .5$

44. Given the following graph of $f(x)$:




(a) Approximate $\int_{-3}^{5} f(x)\,dx$ using 8 equal width rectangles with height given by the right hand endpoint of each rectangle.
Notice that $n = 8$, and $\Delta x = \frac{5-(-3)}{8} = \frac{8}{8} = 1$. Also, we are using Right Hand endpoints.
Therefore, $A \approx 1[f(-2)+f(-1)+f(0)+f(1)+f(2)+f(3)+f(4)+f(5)] = [3+1+2+1+2+2+4+1] = 16$

(b) Use the Trapezoid Rule with $n = 4$ to approximate $\int_{-3}^{5} f(x)\,dx$.
Notice that $n = 4$, so $\frac{b-a}{2n} = \frac{5-(-3)}{2\cdot 4} = \frac{8}{8} = 1$.
Therefore, $A \approx 1[f(-3)+2f(-1)+2f(1)+2f(3)+f(5)] = [0+2(1)+2(1)+2(2)+1] = 9$

45. (a) Use the Trapezoidal Rule with $n = 4$ to approximate $\int_{0}^{\pi} 3\sin x\;dx$
Notice that $n = 4$, so $\frac{b-a}{2n} = \frac{\pi-(0)}{2\cdot 4} = \frac{\pi}{8}$.
Therefore, $A \approx \frac{\pi}{8}[\sin 0 + 2f\sin\frac{\pi}{4} + 2f\sin\frac{\pi}{2} + 2f\sin\frac{3\pi}{4} + f\sin\pi] = \frac{\pi}{8}[0+2\cdot\frac{3\sqrt{2}}{2}+3+2\cdot\frac{3\sqrt{2}}{2}+0] = \frac{\pi}{8}\left(6+6\sqrt{2}\right) \approx 5.6884$

(b) Find the maximum possible error in your approximation from part (a).
Notice that $f'(x) = 3\cos x$ and $f''(x) = -3\sin x$, so $M = 3$.
The maximum error when using the Trapezoidal Rule is given by:
$Error \le \frac{M(b-a)^3}{12n^2} = \frac{3(\pi)^3}{12(4)^2} = \frac{3\pi^3}{192} \approx .48447$

(c) Find the minimum number of rectangles that should be used to guarantee an approximation of $\int_{0}^{\pi} 3\sin x\;dx$ to within 4 decimal places using the Trapezoidal Rule.
Here, we want $Error \le .00005$. Therefore, we need $\frac{3(\pi)^3}{12n^2} \le .00005$ or $\frac{3(\pi)^3}{12(.00005)\le n^2}$. Hence $n^2 \ge 155031.4$ or $n \ge 393.741$, so the minimum number or rectangles is $n = 394$.

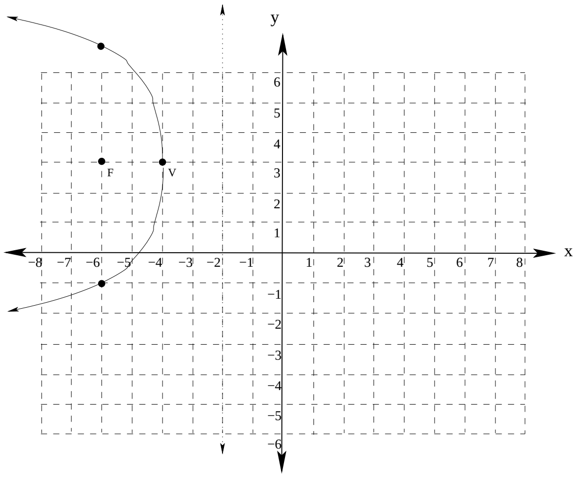
(d) Use the Fundamental Theorem of Calculus to find $\int_{0}^{\pi} 3\sin x\;dx$ exactly. How far off was your estimate? How does the actual error compare to the maximum possible error?
Using the FTC, $\int_{0}^{\pi} 3\sin x\;dx = -3\cos x|_0^\pi = (-3(-1)-(-3)) = 6$.
Our estimate was approximately 5.6884 so the error is about $6 - 5.6884 = .3116$
The estimated maximum error

46. Graphs the following conic sections. Clearly label the main geometric features.

(a) $y^2 - 6y + 8x + 41 = 0$ (oops - slight typo!)
Then $y^2 - 6y = -8x - 41$, so $y^2 - 6y + 9 = -8x - 41 + 9 = -8x - 32$

Therefore $(y-3)^2 = -8(x+4) = 4(-2)(x+4)$. Thus we see that the vertex of this parabola is $V : (-4, 3)$. From the form of the equation, we see that this parabola opens left and $p = -2$, so the focus is $F : (-6, 3)$ and the directrix is $x = -2$

Finally, if we let $x = -6$ and solve using our equation, we see that the points $(-6, -1)$ and $(-6, 7)$ are on the parabola. This gives the graph:

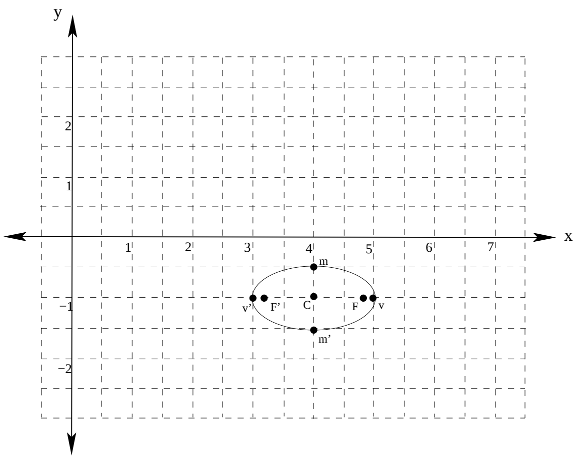


(b) $x^2 + 4y^2 - 8x + 8y + 19 = 0$

By completing the square we obtain: $x^2 - 8x + 16 + 4(y^2 + +1) = -19 + 16 + 4$

Therefore, $(x-4)^2 + 4(y+1)^2 = 1$, or $\frac{x-4)^2}{1} + \frac{(y+1)^2}{\frac{1}{4}} = 1$

Thus we have $C : (4, -1)$, $a = 1$ and $b = \frac{1}{2}$, so $c^2 = 1 - \frac{1}{4} = \frac{3}{4}$, so $c = \frac{\sqrt{3}}{2}$. Hence we have $V : (5, -1)$, $V' : (3, -1)$, $F \approx (3.134, -1)$, $F' : (4.866, -1)$, $M : (4, -.5)$, and $M' : (4, -1.5)$. This gives the graph:

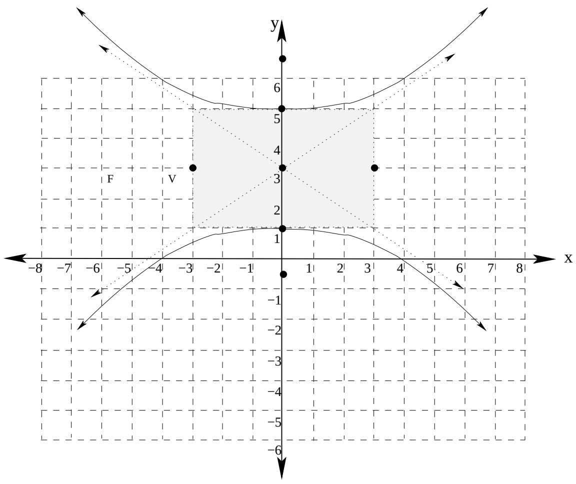


(c) $9y^2 - 4x^2 - 54y + 45 = 0$

By completing the square we obtain: $9(y^2 - 54y + 9) - 4(x^2) = -45 + 81$

Therefore, $9(y-3)^2 - 4x^2 = 36$, or $\frac{y-3)^2}{4} - \frac{(x^2}{9} = 1$

Thus we have $C : (0, 3)$, $a = 2$ and $b = 3$, so $c^2 = 4 + 9 = 13$, so $c = \sqrt{13}$. Hence we have $V : (0, 5)$, $V' : (0, 1)$, $F \approx (0, 6.6056)$, $F' : (0, -.6056)$, $M : (-3, 3)$, and $M' : (3, 3)$. This gives the graph:




47. Find an equation for the conic section with the given features:

(a) A parabola with vertex $(-3, 5)$, and axis parallel to the $x$-axis, and passing through the point $(5, 9)$

Since the directrix is vertical, we have a parabola that opens left/right, so it will have an equation of the form: $(y-k)^2 = 4p(x-h)$. We know that the vertex is $(-3, 5)$ and that $(5, 9)$ is on the parabola, so, substituting these values:

$(9-5)^2 = 4p(5-(-3))$, or $16 = 4p(8)$, so $p = \frac{1}{2}$

Therefore the equation is: $(y-5)^2 = 2(x+3)$

(b) An ellipse with center $(0, 0)$ passing through the points $(2, 3)$ and $(6, 1)$

Since we are looking at an ellipse centered at the origin, is has an equation of the form:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Substituting the known points gives:

$\frac{4}{a^2} + \frac{9}{b^2} = 1$ and $\frac{36}{a^2} + \frac{1}{b^2} = 1$

Therefore, multiplying both sides of these by $a^2b^2$ gives $4b^2 + 9a^2 = a^2b^2$ and $36b^2 + a^2 = a^2b^2$. Thus $4b^2 + 9a^2 = 36b^2 + a^2$, or $8a^2 = 32b^2$, hence $a^2 = 4b^2$, or $a = 2b$ (since both are known to be positive).

Substituting this back into one of our previous equations, we see that $\frac{4}{4b^2} + \frac{9}{b^2} = 1$, or, solving this, $b^2 = 10$, and $a^2 = 40$

Hence the equation is: $\frac{x^2}{40} + \frac{y^2}{10} = 1$

(c) A hyperbola with foci $(0, \pm 3)$ and vertices $(0, \pm 2)$

This is clearly a hyperbola that open up/down and it is symmetric with respect to the both coordinate axes, so the center must be $(0, 0)$. Also, we see that $c = 3$ and $a = 2$. Therefore, $b^2 = c^2 - a^2 = 9 - 4 = 5$

Thus the equation must be: $\frac{y^2}{4} - \frac{x^2}{5} = 1$