

**Recall:**

- The **harmonic series**, defined as  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.
- According to Theorem 11.16, If a series  $\sum a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$

Combining these two pieces of information, we have the following result:

**The  $n$ th Term Test** Given the series  $\sum a_n$ :

- (I) If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , the the series  $\sum a_n$  is divergent.
- (II) If  $\lim_{n \rightarrow \infty} a_n = 0$ , further investigation is needed, as the series might converge, and also might diverge.

**Theorem 11.8** If  $\sum a_n$  and  $\sum b_n$  are series such that  $a_j = b_j$  for every  $j > k$ , where  $k$  is a positive integer, then both series converge or both series diverge.

**Proof:** Since  $a_n = b_n$  for  $j > k$ , then  $\sum_{n=k+1}^{\infty} a_n = \sum_{n=k+1}^{\infty} b_n$ .

Let  $\sum_{n=1}^k a_n = N$  and  $\sum_{n=1}^k b_n = M$ .

Then,  $\sum_{n=1}^{\infty} a_n = N + \sum_{n=k+1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n = M + \sum_{n=k+1}^{\infty} b_n$ .

Hence either both of these converge, or both diverge.

**Theorem 11.19** For any positive integer  $k$ , the series  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$  and  $\sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \dots$  either both converge or both diverge.

**Proof:** Use the same basic argument as above (think of  $\sum b_n$  where  $b_n = 0$  for  $n \leq k$  and  $b_n = a_n$  for  $n > k$ ).

**Theorem 11.20** If  $\sum a_n$  and  $\sum b_n$  are both convergent series with sums  $A$  and  $B$  respectively, then:

- (I)  $\sum a_n + b_n = A + B$
- (II)  $\sum ca_n$  converges and has sum  $cA$  for every real number  $c$ .
- (III)  $\sum a_n - b_n = A - B$

**Theorem 11.21:** If  $\sum a_n$  is a convergent series and  $\sum b_n$  is a divergent series, then  $\sum a_n + b_n$  is divergent.

**Proof:** Suppose that there was an example of a convergent series  $\sum a_n$  and a divergent series  $\sum b_n$  for which the series  $\sum a_n + b_n$  was convergent.

Suppose that  $\sum a_n = A$  and  $\sum a_n + b_n = M$ .

Then, by Theorem 11.20(III),  $\sum [(a_n + b_n) - a_n] = \sum [a_n + b_n] - \sum a_n = M - A$ , so this series converges.

But  $\sum [(a_n + b_n) - a_n] = \sum b_n$ , which diverges by hypothesis.

Since this is clearly preposterous, our assumption that there is an example of a convergent series  $\sum a_n$  and a divergent series  $\sum b_n$  for which the series  $\sum a_n + b_n$  was convergent must be false.

This proves the theorem. [This method of proof is called an “indirect proof”, or a “proof by contradiction”]