Math 262
Trigonometric Substitution Example

Recall:
Trigonometric substitution is a method that allows us to find several types of indefinite and definite integrals that we would otherwise be unable to compute exactly. The main idea behind this technique is taking advantage of the "Pythagorean Identities".

The basic forms of Trigonometric Substitution are as follows:

Expression Form:	**Substitution:**	**Identity:**
1. $a^2 - x^2$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
2. $a^2 + x^2$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
3. $x^2 - a^2$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

A Familiar Example:

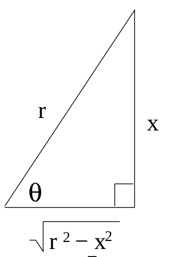
Recall that $y = \sqrt{r^2 - x^2}$ is the equation for a semicircle of radius r. Then, from geometry, we know that $\int_{-r}^{r} \sqrt{r^2 - x^2} = \frac{1}{2}\pi r^2$. We will use trigonometric substitution to evaluate this definite integral analytically:

First note that using symmetry, $\int_{-r}^{r} \sqrt{r^2 - x^2} = \frac{1}{2}\pi r^2 = 2\int_{0}^{r} \sqrt{r^2 - x^2} = \frac{1}{2}\pi r^2$

Let $x = r\sin\theta$. Then $dx = r\cos\theta\, d\theta$.

$$= 2\int_{x=0}^{x=r} \sqrt{r^2 - r^2\sin^2\theta}\, r\cos\theta\ d\theta = 2\int_{x=0}^{x=r} \sqrt{r^2(1 - \sin^2\theta)}\, r\cos\theta\ d\theta = 2\int_{x=0}^{x=r} \sqrt{r^2\cos^2\theta}\, r\cos\theta\ d\theta$$
$$= 2\int_{x=0}^{x=r} r\cos\theta r\cos\theta\ d\theta = 2r^2\int_{x=0}^{x=r} \cos^2\theta\ d\theta = 2r^2\int_{x=0}^{x=r} \frac{1}{2} + \frac{1}{2}\cos(2\theta)\ d\theta$$
$$= 2r^2\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\Big|_{x=0}^{x=r} = 2r^2\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\Big|_{x=0}^{x=r}$$

Where the final step above results from applying the identity: $\sin(2\theta) = 2\sin\theta\cos\theta$
To translate back in terms of x, we build the triangle for our substitution equation:



$$= 2r^2\left[\frac{1}{2}\arcsin\left(\frac{x}{r}\right) + \frac{1}{2}\frac{x}{r}\frac{\sqrt{r^2 - x^2}}{r}\Big|_{0}^{r}\right]$$
$$= 2r^2\left[\frac{1}{2}\arcsin\left(\frac{r}{r}\right) + \frac{1}{2}\frac{r}{r}\frac{\sqrt{r^2 - r^2}}{r} - \frac{1}{2}\arcsin\left(\frac{0}{r}\right) + \frac{1}{2}\frac{0}{r}\frac{\sqrt{r^2 - 0^2}}{r}\right]$$
$$= r^2\arcsin(1) + r^2\frac{\sqrt{0}}{r} - r^2\arcsin(0) - r^2(0)\frac{\sqrt{r^2}}{r} = r^2\left(\frac{\pi}{2}\right) - 0 = \frac{1}{2}\pi r^2$$

Which proves the value we expected from our knowledge of Geometry.