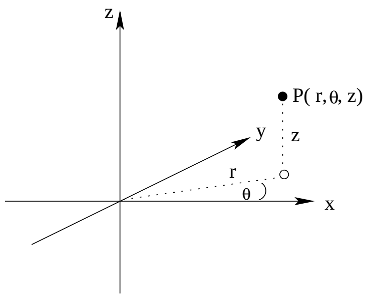
Math 323
Cylindrical and Spherical Coordinates

Cylindrical Coordinates:

Definition: Cylindrical coordinates are an alternate way of describing points in 3-space. Basically, one of the rectangular coordinate planes is replaced by a polar plane (usually the xy-plane, and we will assume this in our descriptions and formulas, but any coordinate plane would do). A point P is then given in terms of the coordinates $P(r, \theta, z)$, where θ is the angle from the positive half of the x-axis, r is the distance from the origin to the projection of P in the xy-plane, and z is the distance from P to the xy-plane.



Definition 17.29: The relationship between a point $P(x, y, z)$ given in polar coordinates and the same point $P(r, \theta, z)$ given in polar coordinates is as follows:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$r^2 = x^2 + y^2$

$\tan\theta = \dfrac{y}{x}.$

To evaluate triple integrals, in cylindrical coordinates, we incorporate the differential for double integrals in polar coordinates $dA = r\,dr\,d\theta$ in order to obtain the differential $dV = dz\,dy\,dx = r\,dz\,dr\,d\theta$

Theorem 17.30 (Fubini for Cylindrical Coordinates):

(I) Let Q be a solid bounded "above" and "below" by a pair of surfaces $k_1(r, \theta)$ and $k_2(r, \theta)$ each sitting "over" a region R in the polar plane and suppose that $f(r, \theta, z)$ is continuous on Q, then

$$\iiint_Q f(r, \theta, z)\ dV = \iint_R \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r, \theta, z)\ dz\,dA$$

(II) Further suppose that the region R is bounded by polar functions $r = g_1(\theta)$ and $r = g_2(\theta)$, for $\alpha \le \theta \le \beta$. Then:

$$\iiint_Q f(r, \theta, z)\ dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r, \theta, z)\, r\,dz\ dr\,d\theta$$

Example: Set up a triple integral in cylindrcial coordinates that gives the volume of the region above $z = \sqrt{x^2 + y^2}$ and below $z = \sqrt{8 - x^2 - y^2}$

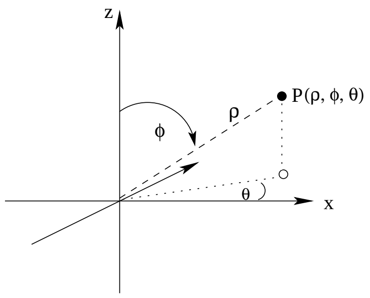
First notice that the intersection in these two surfaces is: $\sqrt{x^2 + y^2} = \sqrt{8 - x^2 - y^2}$ or $x^2 + y^2 = 8 - x^2 - y^2$, so $2x^2 + 2y^2 = 8$. Then $x^2 + y^2 = 4$, so this volume can be thought of as sitting over the circle of radius 2 about the origin in the xy-plane.

Translating everything into cylindrical coordinates, we get $k_1(r, \theta) = \sqrt{r^2} = r$ and $k_2(r, \theta) = \sqrt{8 - r^2}$ describe the two bounding surfaces, and R is given by $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

Thus $V = \displaystyle\int_0^{2\pi} \int_0^2 \int_r^{\sqrt{8-r^2}} 1 \cdot r, dz\,dr\,d\theta$

Spherical Coordinates:

Definition: Spherical coordinates are another way of describing points in 3-space. In this system, a point $P(\rho, \phi, \theta)$ is then given in terms of two angles and a distance. The angle θ is the angle from the positive half of the x-axis to the ray from the origin to the projection of P in the xy-plane. The angle ϕ is the angle of declination from the positive z-axis to the point P, and ρ is the distance between P and the origin.



Definition 17.29: The relationship between a point $P(x, y, z)$ given in polar coordinates and the same point $P(\rho, \phi, \theta)$ given in spherical coordinates is as follows:

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta,$

$z = \rho \cos\phi$

$\rho^2 = x^2 + y^2 + z^2.$

Notice that $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, and $\rho \geq 0$.

To evaluate triple integrals in spherical coordinates, we use the differential for spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

Theorem 17.32 (Fubini for Spherical Coordinates):

(I) Let Q be a solid bounded "inside" and "outside" by the surfaces $\rho = k_1(\phi, \theta)$ and $\rho k_2(\phi, \theta)$ for $\gamma_1 \leq \phi \leq \gamma_2)$ and $\alpha \leq \theta \leq \beta)$. Then:

$$\iiint_Q f(\rho, \phi, \theta) \; dV = \int_\alpha^\beta \int_{\gamma_1}^{\gamma_2} \int_{k_1(\phi,\theta)}^{k_2(\phi,\theta)} f(\rho, \phi, \theta) \, \rho^2 \sin\phi \, d\rho \; d\phi \, d\theta$$

Example: Set up a triple integral in spherical coordinates that gives the volume of the solid bounded below by $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 2$.

First, notice that the region Q is a truncated cone. We must translate the solid region into spherical coordinates. We have $z = 2 = \rho \cos\phi$, so $\rho = \dfrac{2}{\cos\phi}$, or $\rho = 2\sec\phi$. From this, we see that $0 \leq \rho \leq 2\sec\phi$ on this region.

Next, since this surface sits over a circle in the xy-plane, we must have $0 \leq \theta \leq 2\pi$.

Finally, since the cone $z^2 = x^2 + y^2$ has sides bounded by the lines $z = \pm x$ and $z = \pm y$, the sides of the cone sit at 45° angles with both the xy-plane and the positive z-axis. Therefore, $0 \leq \phi \leq \dfrac{\pi}{4}$.

Thus the volume is given by: $V = \displaystyle\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2\sec\phi} 1 \, \rho^2 \sin\phi \, d\rho \; d\phi \, d\theta$