Math 323
Exam 1 Solutions

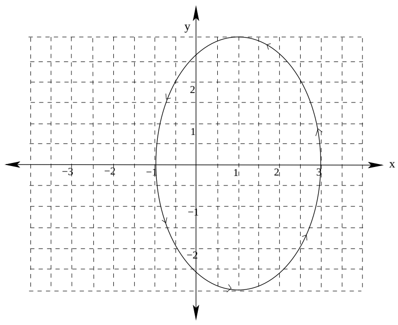
Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Given the curve: $C = \begin{cases} x = 1 + 2\cos t & \text{for } t \in \mathbb{R} \\ y = 3\sin t \end{cases}$

   (a) (5 points) Find an explicit equation for an equation containing the graph of C in terms of x and y.
   First, we solve the parameterizing equations for $\cos t$ and $\sin t$ respectively:
   $x - 1 = 3\cos t$, so $\cos t = \frac{x-1}{2}$ and $\frac{y}{3} = \sin t$
   Then, we use the Pythagorean identity $\sin^2 t + \cos^2 t = 1$ to find our explicit equation:
   $\left(\frac{x-1}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, or $\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$.
   Notice that this is an equation for an ellipse in standard form.

   (b) (5 points) Graph C, indicating the orientation and labeling at least one point on the curve.

t	x	y
0	3	0
$\frac{\pi}{2}$	1	3
π	-1	0
$\frac{3\pi}{2}$	1	-3
2π	3	0



2. Given the curve: $C = \begin{cases} x = t^2 - 2 & \text{for } t \in \mathbb{R} \\ y = t^3 - t \end{cases}$

   (a) (6 points) Find the equation of the tangent line to this curve when $t = 2$.
   Recall that $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-1}{2t}$.
   Then, when $t = 2$, $m = \frac{11}{4}$. Also, when $t = 2$, $x = 2$ and $y = 6$.
   Therefore, the equation for the tangent line to the graph when $t = 2$ is given by: $y - 6 = \frac{11}{4}(x - 2)$.

   (b) (6 points) Determine the concavity of the graph of C when $t = 2$.
   Recall that $\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$.
   Now, $\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{3t^2-1}{2t}\right) = \frac{d}{dt}\left(\frac{3}{2}t - \frac{1}{2}t^{-1}\right) = \frac{3}{2} + \frac{1}{2}t^{-2}$.
   Thus $\frac{d^2y}{dx^2} = \frac{\frac{3}{2}+\frac{1}{2}t^{-2}}{2t}$, which, when $t = 2$ gives:
   $\frac{\frac{3}{2}+\frac{1}{8}}{4} = \frac{13}{32} > 0$, so the graph is concave up at this point.

   (c) (8 points) Set up (But DO NOT evaluate) an integral with respect to t representing the arc length of C for $0 \le t \le 2$.
   $$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_0^2 \sqrt{(2t)^2 + (3t^2 - 1)^2}\, dt$$

3. (5 points) Find a polar equation for the hyperbola $x^2 - y^2 = 9$

Recall: $x = r\cos\theta$ and $y = r\sin\theta$. Then, substituting, we have:

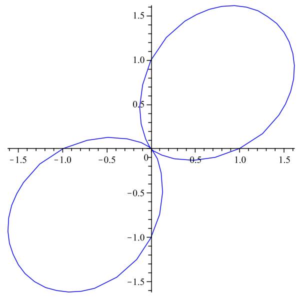
$r^2\cos^2\theta - r^2\sin^2\theta = 9$, or $r^2(\cos^2\theta - \sin^2\theta) = 9$

Then $r^2 = \frac{9}{\cos^2\theta - \sin^2\theta}$, or, using the double angle identity:

$r^2 = \frac{9}{\cos(2\theta)}$ or $r^2 = 9\sec(2\theta)$

4. (10 points) Draw the graph of the polar equation $r = \sin(2\theta) + 1$. Be sure to indicate the orientation and at least two points on the curve.

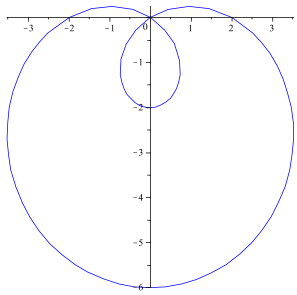
θ	r
0	1
$\frac{\pi}{4}$	2
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	0
π	1
$\frac{5\pi}{4}$	2
$\frac{3\pi}{2}$	1
$\frac{7\pi}{4}$	0
2π	1



Note: the graph should be oriented counterclockwise with two cusp points at the origin.

5. (15 points) Find the area of the inner loop of the polar equation $r = 2 - 4\sin\theta$

Solving $2 - 4\sin\theta = 0$, we have $\sin\theta = \frac{1}{2}$, or $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. The the inner loop is traced out by $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$ [see graph].



Using symmetry, $A = 2\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(2-4\sin\theta)^2\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 16\sin\theta + 16\sin^2\theta\, d\theta$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 16\sin\theta + 16(\frac{1}{2} - \cos(2\theta))\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 16\sin\theta + 8 - 8\cos(2\theta))\, d\theta$$

$$= 12\theta + 16\cos\theta - 4\sin 2\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

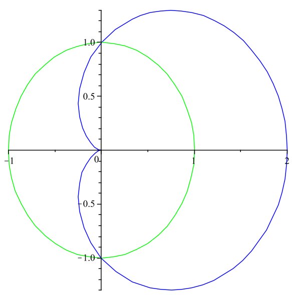
$$= \left(12(\frac{\pi}{2}) + 16(0) - 4(0)\right) - \left(12(\frac{\pi}{6}) + 16(\frac{\sqrt{3}}{2}) - 4(\frac{\sqrt{3}}{2})\right) = 4\pi - 6\sqrt{3}$$

6. (10 points) Set up (But DO NOT evaluate) an integral representing the area inside $r = 1 + \cos\theta$ and outside $r = 1$.

Solving $1 + \cos\theta = 1$, we have $\cos\theta = 0$, or $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Looking carefully at the graph, we see that the region inside $r = 1 + \cos\theta$ and outside $r = 1$ is given when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ [see graph].

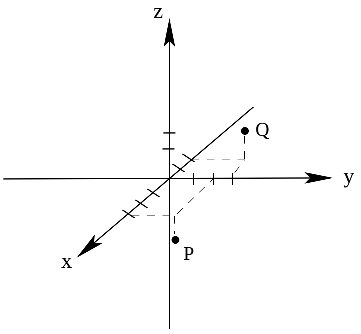
Therefore, to find the area inside $r = 1 + \cos\theta$ and outside $r = 1$ we need only compute:

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[(1+\cos\theta)^2 - 1^2\right]\, d\theta$$



7. Given the points: $P(3, 2, -1)$ and $Q(-2, 3, 2)$

(a) (4 points) Plot P and Q in 3-space.



(b) (4 points) Find $\overrightarrow{PQ}$ and $\|\overrightarrow{PQ}\|$

$\overrightarrow{PQ} = \langle(-2-3),(3-2),(2-(-1))\rangle = \langle -5, 1, 3\rangle$

$\|\overrightarrow{PQ}\| = \sqrt{25+1+9} = \sqrt{35}$

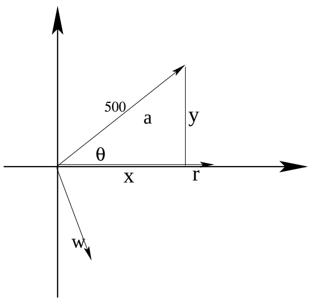
(c) (4 points) Find a vector with magnitude 7 in the opposite direction as $\overrightarrow{PQ}$.

$\vec{v} = -7\frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{\langle 35,-7,-21\rangle}{\sqrt{35}} = \langle\sqrt{35}, -\frac{7}{\sqrt{35}}, -\frac{21}{\sqrt{35}}\rangle$

(d) (4 points) Find an equation for the sphere centered at Q and containing P.

$(x+2)^2 + (y-3)^2 + (z-2)^2 = 35$

8. (8 points) Suppose the thrust of an airplane's engine produces a speed on 500mph in still air and wind velocity is given by $\langle 20, -80\rangle$. Find the direction the plane should head in order to fly due east. Also find the speed at which the plane will travel this course.



Let $\vec{w} = \langle 20, -80\rangle$ be the wind vector, $\vec{a} = \langle x, y\rangle$ be the vector of the airplane in still air, and $\vec{a} + \vec{w} = \vec{r} = \langle 0, s\rangle$ be the resultant vector. Let θ be the angle between $\vec{a}$ and the positive x-axis.

Since the speed of the airplane is 500 mph, then $x^2 + y^2 = 500^2$. Also, since the airplane ends up heading due east, then $y = 80$. Therefore, $x = \sqrt{500^2 - 80^2} = \sqrt{243,600} = 20\sqrt{609}$.

Thus $\vec{a} = \langle 20\sqrt{609}, 80\rangle$, and so $\theta = \arctan\frac{80}{20\sqrt{609}} \approx 9.2°$.

Therefore, the airplane should travel at the heading 80.8° East of North.

Finally, the speed of the plane $s = 20 + 20\sqrt{609} \approx 513.6$ mph.