Math 323
Exam 3 - Solutions **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. (a) (10 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+4xy+y^2}{x^2-2xy+y^2}$ does not exist.

   First, consider the limit along $x = 0$:
   $$\lim_{(0,y)\to(0,0)} \frac{0^2+4(0)y+y^2}{0^2-2(0)y+y^2} = \lim_{(0,y)\to(0,0)} \frac{y^2}{y^2} = 1$$

   Next, consider the limit along $x = 2y$:
   $$\lim_{(2y,y)\to(0,0)} \frac{(2y)^2+4(2y)y+y^2}{(2y)^2-2(2y)y+y^2} = \lim_{(2y,y)\to(0,0)} \frac{13y^2}{y^2} = 13$$

   Since the limit along these two paths disagree, we know that this limit does not exist.
   Notes:
   i) Since this expression is undefined whenever $x = y$, we cannot compute the limit along this path.
   ii) Any other linear path $x = my$ could be used to show the limit does not exist.

   (b) (8 points) Describe the set of points at which the function $f(x,y) = \dfrac{x^2+4xy+y^2}{x^2-2xy+y^2}$ is *discontinuous*.

   This function is discontinuous whenever $x^2-2xy+y^2 = 0$. That is, when $(x-y)^2 = 0$. Hence the this function is discontinuous along the line $x = y$.

2. Given the equation $x^2y - 2xy^2 + 4xyz^2 - 3 = 0$:

   (a) (8 points) Use implicit differentiation to find $\dfrac{\partial z}{\partial y}$

   Recall that $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{x^2-4xy+4xz^2}{8xyz} = -\dfrac{x-4y+4z^2}{8yz}$

   (b) (8 points) Find an equation for the tangent plane to this surface at the point (1,1,1).

   $\nabla F = \langle 2xy - 2y^2 + 4yz^2, x^2 - 4xy + 4xz^2, 8xyz\langle$
   Then $\nabla F(1,1,1) = \langle 4, 1, 8\rangle$
   Hence the plane has equation $4(x-1) + (y-1) + 8(z-1) = 0$ or $4x + y + 8z - 13 = 0$

3. Let $w = f(x, y) = x^2 - 2xy + 4y$

   (a) (8 points) Find an expression for Δw.

   $\Delta w = f(x + \Delta x, y + \Delta y) - f(x, y) = (x + \Delta x)^2 - 2(x + \Delta x)(y + \Delta y) + 4(y + \Delta y) - (x^2 - 2xy + 4y)$
   $= x^2 + 2x\Delta x + (\Delta x)^2 - 2(xy + x\Delta y + y\Delta x + \Delta x\Delta y) + 4y + 4\Delta y - x^2 + 2xy - 4y$
   $= x^2 + 2x\Delta x + (\Delta x)^2 - 2xy - 2x\Delta y - 2y\Delta x - 2\Delta x\Delta y + 4y + 4\Delta y - x^2 + 2xy - 4y$
   $= 2x\Delta x + (\Delta x)^2 - 2x\Delta y - 2y\Delta x - 2\Delta x\Delta y + 4\Delta y$

   (b) (6 points) Find the terms ε_1 and ε_2 in the standard decomposition for Δw

   Notice that $f_x = 2x - 2y$ and $f_y = -2x + 4$. Therefore, if we put Δw into the form:

   $\Delta w = f_x\Delta x + f_y\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$, we get
   $\Delta w = (2x - 2y)\Delta x + (-2x + 4)\Delta y + (\Delta x)(\Delta x) + (-2\Delta x)\Delta y$

   Therefore, $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = -2\Delta x$
   (Well, this is the simplest way to write this decomposition.)

   (c) (6 points) Use dw to approximate Δw as the input for $w = f(x, y)$ changes from $(1, 1)$ to $(0.9, 1.2)$

   Notice that $\Delta x = -0.1$, $\Delta y = 0.2$, and $P(1, 1)$
   Then $dw = f_x\Delta x + f_y\Delta y = (2x - 2y)\Delta x + (-2x + 4)\Delta y$
   $= (2(1) - 2(1))(-0.1) + (-2(1) + 4)(0.2) = 2(0.2) = 0.4$

4. (12 points) Suppose $w = f(x, y)$ where $x = s^2t$ and $y = s - 2t$. Also suppose that $f_x(x, y) = x^2 + e^y$ and $f_y(x, y) = x^2e^y$. Find the value of $\dfrac{\partial w}{\partial s}$ when $s = 2$ and $t = 1$.

   Recall that $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} = \left(x^2 + e^y\right)(2st) + \left(x^2e^y\right)(1)$
   Notice that when $s = 2$ and $t = 1$, $x = 2^2(1) = 4$ and $y = 2 - 2(1) = 0$
   Therefore, evaluating gives: $\dfrac{\partial w}{\partial s} = (4^2 + e^0)(2(2)(1)) + (4^2(e^0))(1) = 17(4) + (16) = 84.$

5. Given that $f(x,y) = \sqrt{x^2 + 5xy + 3y^2}$

   (a) (10 points) Find the derivative of f at the point (2,2) and in the direction from (2,2) to (3,1).

   First notice that $f_x = \dfrac{1}{2}(x^2 + 5xy + 3y^2)^{-\frac{1}{2}}(2x + 5y) = \dfrac{2x + 5y}{2\sqrt{x^2 + 5xy + 3y^2}}$

   Similarly, $f_y = \dfrac{1}{2}(x^2 + 5xy + 3y^2)^{-\frac{1}{2}}(5x + 6y) = \dfrac{5x + 6y}{2\sqrt{x^2 + 5xy + 3y^2}}$

   Then $f_x(2,2) = \frac{4+10}{2(\sqrt{36})} = \frac{14}{12} = \frac{7}{6}$ and $f_y(2,2) = \frac{10+12}{2(\sqrt{36})} = \frac{22}{12} = \frac{11}{6}$

   Thus $\nabla f(2,2) = \langle \frac{7}{6}, \frac{11}{6} \rangle$

   Next, notice $\vec{v} = \langle 1, -1 \rangle$, so $\vec{u} = \frac{\langle 1,-1 \rangle}{\sqrt{1+1}} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$.

   Hence $D_{\vec{u}} f(2,2) = \langle \frac{7}{6}, \frac{11}{6} \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = \frac{7}{6\sqrt{2}} - \frac{11}{6\sqrt{2}} = -\frac{4}{6\sqrt{2}} = -\frac{2}{3\sqrt{2}} = -\frac{\sqrt{2}}{3}$.

   (b) (8 points) Find the direction and magnitude of the maximum rate of change of $f(x,y)$ at the point (1,1).

   Recall that the maximum rate of change occurs in the direction of the gradient:
   Now $f_x(1,1) = \frac{2+5}{2(\sqrt{9})} = \frac{7}{6}$ and $f_y(1,1) = \frac{5+6}{2(\sqrt{9})} = \frac{11}{6}$

   So $\nabla f(1,1) = \langle \frac{7}{6}, \frac{11}{6} \rangle$ is the direction of the maximum rate of change.

   Thus the maximum rate of change is $\|\nabla f(1,1)\| = \sqrt{\dfrac{49+121}{36}} = \dfrac{\sqrt{170}}{6}$

6. (20 points) Find all the critical points of $f(x,y) = x^2 - 2x + xy^2 + y^2$, and classify them using the Discriminant.

Notice that $f_x = 2x - 2 + y^2$ and $f_y = 2xy + 2y$. Since both of these are continuous, all critical points occur when the partial derivatives are both zero. We now solve to find the all critical points:

If $f_y = 2xy + 2y = 0$, then $2y(x+1) = 0$, so either $y = 0$ or $x = -1$.

If $y = 0$ then $f_x = 2x - 2 + 0^2 = 0$, or $2x = 2$, so $x = 1$. Then $(1,0)$ is a critical point.

If $x = -1$ then $f_x = 2(-1) - 2 + y^2 = 0$, or $-4 + y^2 = 2$, or $y^2 = 4$, so $y = \pm 2$. Then $(-1,2)$ and $(-1,-2)$ are critical points.

Next, to compute the discriminant for each critical point, $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2$, we compute the second partials:

$f_{xx} = 2$, $f_{yy} = 2x + 2$, and $f_{xy} = f_{yx} = 2y$.

Therefore:

$D(1,0) = (2)(4) - (0)^2 = 8 > 0$. Since $f_{xx}(1,0) > 0$, we know we have a local minimum at $(1,0)$.

$D(-1,2) = (2)(0) - (4)^2 = 16 < 0$. So we know we have a saddle point at $(-1,2)$.

$D(-1,-2) = (2)(0) - (4)^2 = 16 < 0$. So we know we have a saddle point at $(-1,-2)$.