

1. Cascade Container Company produces a cardboard shipping crate at three different plants in amounts  $x$ ,  $y$ , and  $z$ , respectively. Their annual revenue is  $R(x, y, z) = 2xyz^2$ . The company needs to produce 1000 crates annually. How many crates should they produce at each plant in order to maximize their revenue?

We will apply the method of La Grange to the function  $R(x, y, z) = 2xyz^2$  subject to the constraint  $g(x, y, z) = x + y + z = 1,000$ .

Notice that  $R_x = 2yz^2$ ,  $R_y = 2xz^2$ ,  $R_z = 4xyz$ , and  $g_x = g_y = g_z = 1$ .

Therefore, we have  $2yz^2 = \lambda = 2xz^2 = 4xyz$ . Hence  $z = 0$ , or  $x = y$  and  $2yz^2 = 4y^2z$ , so  $z = 2y$ .

If  $z = 0$ , then  $R(x, y, z) = 0$

If  $x = y$  and  $z = 2y$ , then  $y + y + 2y = 1,000$ , or  $4y = 1000$ , so  $x = y = 250$  and  $z = 500$ .

In this case,  $R(x, y, z) = 2(250)(250)(500)^2 = 31,250,000,000$  units of revenue (probably \$, but not specified)

2. Use Lagrange multipliers to maximize  $f(x, y) = 4x^2y$  subject to the constraint  $x^2 + y^2 = 3$ .

We will apply the method of La Grange to the function  $f(x, y) = 4x^2y$  subject to the constraint  $g(x, y) = x^2 + y^2 - 3 = 0$ .

Notice that  $f_x = 8xy$ ,  $f_y = 4x^2$ ,  $g_x = 2x$ , and  $g_y = 2y$ . Also, we must have that  $\nabla f = \lambda \nabla g$ .

Therefore,  $8xy = 2\lambda x$ , so either  $x = 0$  or  $4y = \lambda$ . [Notice that if  $x = 0$ ,  $f(x, y) = 0$ ]

Using the other pair of partials,  $4x^2 = \lambda 2y$ , or, substituting,  $4x^2 = 8y^2$ , or  $x^2 = 2y^2$ .

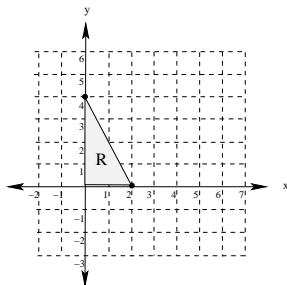
We use this to substitute into the constraint, yielding:  $2y^2 + y^2 = 3$ , or  $3y^2 = 3$ , so  $y = \pm 1$

But then  $x^2 = 2$ , so  $x = \pm\sqrt{2}$ .

Finally,  $f(\pm\sqrt{2}, 1) = 4(2)(1) = 8$ , and  $f(\pm\sqrt{2}, -1) = 4(2)(-1) = -8$ .

Hence the maximum of  $f(x, y)$  subject to  $x^2 + y^2 = 3$  is 8, which occurs at  $(\sqrt{2}, 1)$  and  $(-\sqrt{2}, 1)$ .

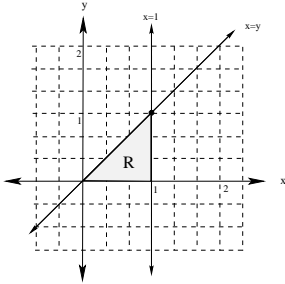
3. Evaluate  $\iint_R (y + x) dA$ , where  $R$  is the region bounded by  $x = 0$ ,  $y = 0$ , and  $2x + y = 4$ .



$$\begin{aligned} \int_0^2 \int_0^{4-2x} (y + x) dy dx &= \int_0^2 \left. \frac{1}{2}y^2 + xy \right|_0^{4-2x} dx \\ &= \int_0^2 \frac{1}{2}(4-2x)^2 + x(4-2x) - 0 dx = \int_0^2 \frac{1}{2}(16 - 16x + 4x^2) + 4x - 2x^2 dx = \int_0^2 8 - 4x dx \\ &= 8x - 2x^2 \Big|_0^2 = 16 - 8 = 8 \end{aligned}$$

4. For each of the double integrals given below, first graph the region of integration. Then reverse the order of integration for the iterated integral and then evaluate the integral exactly.

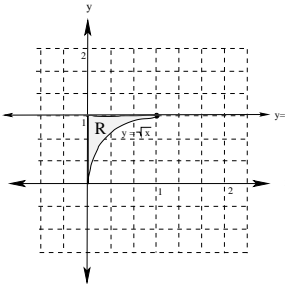
(a)  $\int_0^1 \int_y^1 3xe^{x^3} dx dy$



$$= \int_0^1 \int_0^x 3xe^{x^3} dy dx = \int_0^1 3xye^{x^3} \Big|_0^x dx = \int_0^1 3x^2e^{x^3} dx$$

$$= e^{x^3} \Big|_0^1 = e^1 - e^0 = e - 1$$

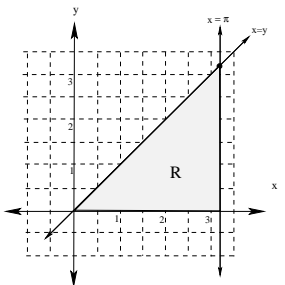
(b)  $\int_0^1 \int_{\sqrt{x}}^1 \frac{3}{4+y^3} dy dx$



$$= \int_0^1 \int_0^{y^2} \frac{3}{4+y^3} dx dy = \int_0^1 \frac{3x}{4+y^3} \Big|_0^{y^2} dy = \int_0^1 \frac{3y^2}{4+y^3} dy$$

$$= \ln |4+y^3| \Big|_0^1 = \ln 5 - \ln 4 = \ln \frac{5}{4}$$

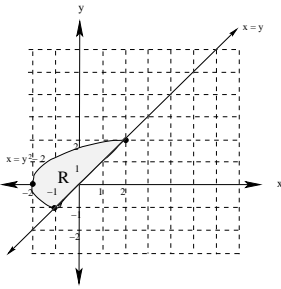
(c)  $\int_0^\pi \int_y^\pi \left(1 - \frac{\cos x}{x}\right) dx dy$



$$= \int_0^\pi \int_0^x 1 - \frac{\cos x}{x} dy dx = \int_0^\pi y - y \frac{\cos x}{x} \Big|_0^x dx = \int_0^\pi x - \cos x dx$$

$$= \frac{1}{2}x^2 - \sin x \Big|_0^\pi = \frac{\pi^2}{2} - 0 = \frac{\pi^2}{2}$$

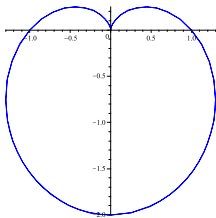
5. Express the volume of the solid bounded by the curves  $z = x + 2$ ,  $z = 0$ ,  $x = y^2 - 2$  and  $x = y$  as a double integral in rectangular coordinates. Also, sketch the region in the plane for the integration. DO NOT EVALUATE THE INTEGRAL.



Notice that the points of intersection for the region in the plane occur when  $y = y^2 - 2$ , or when  $y^2 - y - 2 = 0$ . That is, when  $(y - 2)(y + 1) = 0$ , so when  $y = 2$  or  $y = -1$ . Therefore, the volume is given by:

$$\int_{-1}^2 \int_{y^2-2}^y x + 2 \, dx \, dy$$

6. Evaluate  $\iint_R x \, dA$  where  $R$  is the region in the polar plane bounded by  $r = 1 - \sin \theta$ .



Converting everything to polar coordinates,  $x = r \cos \theta$  and  $r = 1 - \sin \theta$  is the cardioid graphed above.

$$\text{Then we have } \int_0^{2\pi} \int_0^{1-\sin \theta} r \cos \theta r \, dr \, d\theta = \int_0^{2\pi} \int_0^{1-\sin \theta} r^2 \cos \theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} r^3 \cos \theta \Big|_0^{1-\sin \theta} d\theta = \frac{1}{3} \int_0^{2\pi} (1 - \sin \theta)^3 \cos \theta \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} (1 - 3 \sin \theta + 3 \sin^2 \theta - \sin^3 \theta) \cos \theta \, d\theta = \frac{1}{3} \left[ \cos \theta - \frac{3}{2} \sin^2 \theta + \sin^3 \theta - \frac{1}{4} \sin^4 \theta \Big|_0^{2\pi} \right] = 0$$

[Note that this is not a surprise as  $f(x, y) = x$  is odd and the region  $R$  is symmetric with respect to the  $y$ -axis]

7. Calculate the mass of a lamina that occupies the plane region  $R$  bounded by the curve  $(x - 1)^2 + y^2 = 1$  with density function  $\rho(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$ .

Clearly, this problem is best represented in polar coordinates. First,  $\rho(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$  becomes  $\rho(r, \theta) = \frac{1}{\sqrt{r^2}} = \frac{1}{r}$ .

Next, the region in the plane is given by:  $x^2 - 2x + 1 + y^2 = 1$ , or  $r^2 - 2r \cos \theta = 0$ . Thus we have  $r^2 = 2r \cos \theta$ , or  $r = 2 \cos \theta$ .

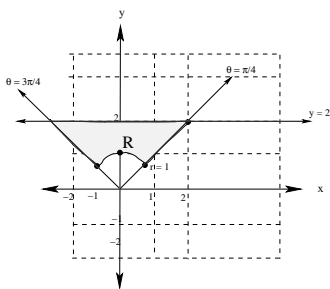
$$\text{Therefore, using symmetry, we have mass} = 2 \int_0^{\pi/2} \int_0^{\cos \theta} \frac{1}{r} r \, dr \, d\theta = 2 \int_0^{\pi/2} 2 \cos \theta \, d\theta$$

$$= 4 \sin \theta \Big|_0^{\pi/2} = 4(1) - 0 = 4$$

8. Sketch the region of integration for  $\int_{\pi/4}^{3\pi/4} \int_1^{2/\sin\theta} r \, dr \, d\theta$ .

Notice that we have  $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ . Also,  $r = 1$  is the circle of radius 1 centered at the origin, and if  $r = \frac{2}{\sin\theta}$ , then  $r \sin\theta = 2$  or  $y = 2$ .

Therefore the region of integration is as follows:



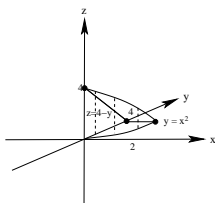
9. Find the surface area of the surface  $S$  where  $S$  is first octant portion of the hyperbolic paraboloid  $z = x^2 - y^2$  that is inside the cylinder  $x^2 + y^2 = 1$ .

Recall that  $S = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dA$ . Here,  $z = f(x, y) = x^2 - y^2$ , so  $f_x = 2x$  and  $f_y = -2y$

So we have  $S = \iint_R \sqrt{(2x)^2 + (-2y)^2 + 1} \, dA = \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA$  where  $R$  is portion of the cylinder  $x^2 + y^2 = 1$  in the first octant (so  $x \geq 0$ ,  $y \geq 0$ , and  $z \geq 0$ ).

$$\begin{aligned} \text{Converting to polar coordinates gives: } & \int_0^{\pi/4} \int_0^1 \sqrt{4r^2 + 1} r \, dr \, d\theta = \int_0^{\pi/4} \frac{2}{3} \cdot \frac{1}{8} (4r^2 + 1)^{\frac{3}{2}} \Big|_0^1 \, d\theta \\ & = \frac{1}{12} \int_0^{\pi/4} 5^{\frac{3}{2}} - 1 \, d\theta = \frac{1}{12} \frac{\pi}{4} (5^{\frac{3}{2}} - 1) = \frac{\pi}{48} (5^{\frac{3}{2}} - 1). \end{aligned}$$

10. Let  $I = \int_0^2 \int_{x^2}^4 \int_0^{4-y} x + yz \, dz \, dy \, dx$ . Sketch the solid  $Q$  over which the iterated integral takes place, and rewrite the iterated integral in the order  $dx \, dz \, dy$ . DO NOT EVALUATE THE INTEGRAL.



Notice that we have  $0 \leq z \leq 4 - y$ ,  $x^2 \leq y \leq 4$ , and  $0 \leq x \leq 2$

Then we also have  $0 \leq x \leq \sqrt{y}$ ,  $0 \leq z \leq 4 - y$ , and  $0 \leq y \leq 4$

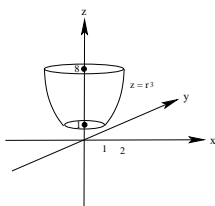
Thus the integral in the required order is:  $\int_0^4 \int_0^{4-y} \int_0^{\sqrt{y}} x + yz \, dx \, dz \, dy$

11. Find the mass of the cylinder of radius 3 between  $z = 0$  and  $z = 4$  if the density at the point  $(x, y, z)$  is given by  $\delta(x, y, z) = z + \sqrt{x^2 + y^2}$ .

This mass integral is best represented in cylindrical coordinates:

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^3 \int_0^4 (z + r) r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^3 \int_0^4 zr + r^2 \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^3 \frac{1}{2} z^2 r + r^2 z \Big|_0^4 \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^3 8r + 4r^2 \, dr \, d\theta \\ &= \int_0^{2\pi} 4r^2 + \frac{4}{3} r^3 \Big|_0^3 \, d\theta = \int_0^{2\pi} 36 + 36 \, d\theta = 72\theta \Big|_0^{2\pi} = 144\pi \end{aligned}$$

12. Let  $Q$  be the region between  $z = (x^2 + y^2)^{3/2}$  and  $z = 1$ , and inside  $x^2 + y^2 = 4$ . Sketch the region  $Q$ , and then write  $\iiint_Q \sqrt{x^2 + y^2} e^z dV$  as an integral in the best (for this example) 3-dimensional coordinate system. DO NOT EVALUATE THE INTEGRAL.



Given the form of the solid region and the function, cylindrical coordinates is the best system to use to express this integral.

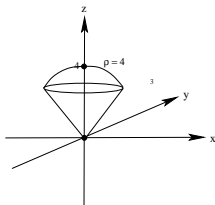
Converting the top and bottom surfaces, we have  $z = (r^2)^{3/2} = r^3$  and  $z = 1$  [this was a typo on the original handout-sorry!]

Notice that the intersection between these two surfaces is:  $(x^2 + y^2)^{3/2} = 1$ , or  $x^2 + y^2 = 1$ , the circle of radius 1 about the origin.

The region in the plane is all points between the circle of radius one and the circle of radius 2 both centered at the origin, and the integrand becomes  $re^z$ .

Thus the integral is: 
$$\int_0^{2\pi} \int_1^2 \int_1^{r^3} (re^z) r dz dr d\theta = \int_0^{2\pi} \int_1^2 \int_1^{r^3} r^2 e^z dz dr d\theta$$

13. Set up a triple integral in spherical coordinates for the volume  $V$  of the region between  $z = \sqrt{3x^2 + 3y^2}$  and the sphere  $x^2 + y^2 + z^2 = 16$ . Be sure to include a sketch of the region, and DO NOT EVALUATE THE INTEGRAL.



Translating  $z = \sqrt{3x^2 + 3y^2}$ , we have  $z^2 = 3x^2 + 3y^2$ , or  $4z^2 = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$

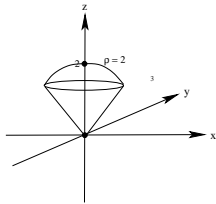
Then  $4(\rho \cos \phi)^2 = 3\rho^2$ , or  $4\rho^2 \cos^2 \phi = 3\rho^2$ . Thus we have  $4 \cos^2 \phi = 3$ , or  $\cos^2 \phi = \frac{3}{4}$ .

Hence  $\cos \phi = \pm \frac{\sqrt{3}}{2}$ . Since we only want values between 0 and  $\pi$ , we take  $\phi = \frac{\pi}{6}$ , [a 30° cone].

Notice that  $x^2 + y^2 + z^2 = 16$  is the sphere of radius 4 about the origin, and has the form  $\rho = 4$ .

Therefore, the integral has the form 
$$\int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^4 \rho^2 \sin \phi d\rho d\phi d\theta$$

14. (a) Let  $Q$  be the region between  $z = \sqrt{3x^2 + 3y^2}$  and  $z = \sqrt{4 - x^2 - y^2}$ . Sketch the region  $Q$



- (b) Set up, but do not evaluate a triple integral in rectangular coordinates that gives the volume of  $Q$ .

Using symmetry, 
$$\iiint_Q 1 dV = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} 1 dz dy dx$$

- (c) Set up, but do not evaluate a triple integral in cylindrical coordinates that gives the volume of  $Q$ .

Again using symmetry, 
$$\iiint_Q 1 dV = 4 \int_0^{\frac{\pi}{2}} \int_0^1 \int_{\sqrt{3r}}^{\sqrt{4-r^2}} 1 r dz dr d\theta$$

(d) Set up, but do not evaluate a triple integral in spherical coordinates that gives the volume of  $Q$ .

We need to figure out the angle of declination for the cone. Starting with  $z^2 = 3x^2 + 3y^2$ , we see that  $4z^2 = 3x^2 + 3y^2 + 3z^2$ , or  $4z^2 = 3\rho^2$ .

Therefore, since  $z = \rho \cos \phi$ ,  $4\rho^2 \cos^2 \phi = 3\rho^2$ . Hence  $4 \cos^2 \phi = 3$ , or  $\cos^2 \phi = \frac{3}{4}$ .

Thus  $\cos \phi = \pm \frac{\sqrt{3}}{2}$ . The solution to this in the first quadrant is  $\phi = \frac{\pi}{6}$ .

So we have  $\iiint_Q 1 \, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(e) Pick one of the triple integrals you found above and evaluate it in order to find the volume of  $Q$  exactly.

We will use the spherical integral:  $\int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \frac{1}{3} \rho^3 \sin \phi \Big|_0^2 \, d\phi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \frac{8}{3} \sin \phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} -\cos \phi \Big|_0^{\frac{\pi}{6}} \, d\theta$$

$$\frac{8}{3} \int_0^{2\pi} -\cos \phi \Big|_0^{\frac{\pi}{6}} \, d\theta = \frac{8}{3} \int_0^{2\pi} -\left(\frac{\sqrt{3}}{2} - 1\right) \, d\theta = \frac{8}{3} \left(\frac{1 - \sqrt{3}}{2}\right) \theta \Big|_0^{2\pi} = \frac{8}{3} (2\pi) \left(1 - \frac{\sqrt{3}}{2}\right) = \frac{16\pi}{3} \left(1 - \frac{\sqrt{3}}{2}\right)$$