Math 323
Exam 4 - Solutions **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. (16 points) Use Lagrange multipliers to maximize $f(x, y) = 4x^2y$ subject to the constraint $x^2 + y^2 = 3$.

   This problem should have looked oddly familiar since it was taken from the practice problems.

   We will apply the method of La Grange to the function $f(x, y) = 4x^2y$ subject to the constraint $g(x, y) = x^2+y^2-3 = 0$.

   Notice that $f_x = 8xy$, $f_y = 4x^2$, $g_x = 2x$, and $g_y = 2y$. Also, we must have that $\nabla f = \lambda \nabla g$.

   Therefore, $8xy = 2\lambda x$, so either $x = 0$ or $4y = \lambda$. [Notice that if $x = 0$, $f(x, y) = 0$]

   Using the other pair of partials, $4x^2 = \lambda 2y$, or, substituting, $4x^2 = 8y^2$, or $x^2 = 2y^2$.

   We use this to substitute into the constraint, yielding: $2y^2 + y^2 = 3$, or $3y^3 = 3$, so $y = \pm 1$

   But then $x^2 = 2$, so $x = \pm\sqrt{2}$.

   Finally, $f(\pm\sqrt{2}, 1) = 4(2)(1) = 8$, and $f(\pm\sqrt{2}, -1) = 4(2)(-1) = -8$.

   Hence the maximum of $f(x, y)$ subject to $x^2 + y^2 = 3$ is 8, which occurs at both $(\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$.

2. (16 points) A lamina occupies the plane region R bounded by the curve $x^2 + y^2 = 4$ and has density function $\rho(x, y) = \sqrt{x^2 + y^2 + 1}$ where the units are in g/cm^2. Find the mass if this lamina.

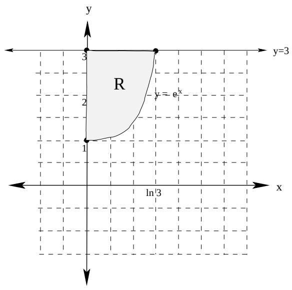
   Clearly, the correct coordinate system to use for this problem is polar coordinates. Notice that the region of integration is the circle of radius 2 centered at the origin, and the density function, when translated into polar form is $\rho(r, \theta) = \sqrt{r^2 + 1}$. Then we can find the mass using the following integral:

   $$\int_0^{2\pi}\int_0^2 \sqrt{r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{2}{3}\left(r^2+1\right)^{\frac{3}{2}} \frac{1}{2}\Bigg|_0^2 d\theta$$

   $$= \int_0^{2\pi} \frac{1}{3}\left[(4+1)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] d\theta$$

   $$= \frac{1}{3}\left[(4+1)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]\theta\Bigg|_0^{2\pi} = \frac{2\pi}{3}\left(5^{\frac{3}{2}} - 1\right) \text{ grams.}$$

3. (16 points) Reverse the order of integration for the double integral $\int_1^3 \int_0^{\ln y} f(x,y)\,dx\,dy$

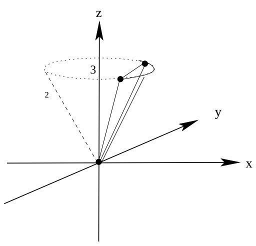


Notice that if $x = \ln y$, then $y = e^x$. Also, we have $0 \le x \le \ln 3$. Then the integral becomes:

$$\int_0^{\ln 3} \int_{e^x}^{3} f(x,y)\,dy\,dx$$

4. Let Q be the region between $z = 3$ and $z = \sqrt{x^2 + y^2}$ in the first octant.

(a) (7 points) Sketch the region Q.



(b) (10 points) Set up a triple integral which gives $I = \iiint_Q x\,dV$ in cylindrical coordinates.
DO NOT EVALUATE THE INTEGRAL.

From the graph above, we see that we are integrating over a quarter circle of radius 3, with top surface $z = 3$ and bottom surface $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$ Also, $x = r\cos\theta$. Thus we have the following integral in cylindrical coordinates:

$$\int_0^{\frac{\pi}{2}} \int_0^3 \int_r^3 r\cos\theta\, r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^3 \int_r^3 r^2 \cos\theta\,dz\,dr\,d\theta$$

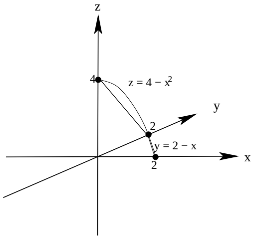
(c) (10 points) Set up a triple integral which gives $I = \iiint_Q x\,dV$ in spherical coordinates.
DO NOT EVALUATE THE INTEGRAL.

Again from the graph above, we see that we are integrating inside a $\frac{1}{4}$-cone with angle of declination $\frac{\pi}{4}$ and bounded on top by the surface $z = 3$. In spherical coordinates, is $z = \rho\cos\phi = 3$, then $\phi = \frac{3}{\cos\phi} = 3\sec\phi$. Also, $dV = \rho^2 \sin\phi$, and $x = \rho\sin\phi\cos\theta$. Thus we have the following integral in spherical coordinates:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{3\sec\phi} \rho\sin\phi\cos\theta\rho^2\sin\theta\,d\rho\,d\phi\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{3\sec\phi} \rho^3 \sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$

5. Let a volume integral be defined by $V = \int_0^2 \int_0^{4-x^2} \int_0^{2-x} dy\, dz\, dx$

(a) (6 points) Sketch the solid Q whose volume is given by this iterated integral.



(b) (12 points) Express this integral in rectangular coordinates in the order $dz\, dy\, dx$.
Looking carefully at the graph above, we see that we have $0 \leq x \leq 2$, $0 \leq y \leq 2 - x$, and $0 \leq z \leq 4 - x^2$
Then we have the following integral when we reorder:

$$= \int_0^2 \int_0^{2-x} \int_0^{4-x^2} dz\, dy\, dx$$

(c) (12 points) Evaluate this integral.

$$= \int_0^2 \int_0^{2-x} \int_0^{4-x^2} dz\, dy\, dx = \int_0^2 \int_0^{2-x} 4 - x^2 dy\, dx = \int_0^2 (4 - x^2)y\Big|_{2-x}^{0} dx$$

$$= \int_0^2 (4 - x^2)(2 - x)\, dx = \int_0^2 8 - 4x - 2x^2 + x^3\, dx = 8x - 2x^2 - \frac{2}{3}x^2 + \frac{1}{4}x^2\Big|_0^2$$

$$= 16 - 8 - \frac{16}{3} + 4 = 12 - \frac{16}{3} = \frac{20}{3}.$$