Math 323
Mass, Moments, and Center of Mass

1. **Mass:**

   (a) 2D case: Suppose that L is 2-dimensional lamina in the shape of a region R in the plane and that $\delta(x,y)$ is a function that gives the density per unit area at each point of R. Then the mass of L is given by:
   $m = \iint_R \delta(x,y)\,dA$

   (b) 3D case: Suppose that L is 3-dimensional solid in the shape of a solid region Q in space and that $\delta(x,y,z)$ is a function that gives the density per unit of volume at each point of Q. Then the mass of L is given by:
   $m = \iiint_Q \delta(x,y,z)\,dV$

2. **First Moments and the Center of Mass:**

   (a) 2D case: Suppose that L is 2-dimensional lamina in the shape of a region R in the plane and that $\delta(x,y)$ is a function that gives the density per unit area at each point of R. Then the first moment of L with respect to the x and y axes are:

   $M_x = \iint_R y\delta(x,y)\,dA$ and $M_y = \iint_R x\delta(x,y)\,dA$

   The center of mass of the lamina L is $(\overline{x},\overline{y})$, where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$

   (b) 3D case: Suppose that L is 3-dimensional solid in the shape of a solid region Q in space and that $\delta(x,y,z)$ is a function that gives the density per unit of volume at each point of Q. Then the first moment of L with respect to the xy, xz, and yz planes are:

   $M_{xy} = \iiint_Q z\delta(x,y,z)\,dV$, $M_{xz} = \iiint_Q y\delta(x,y,z)\,dV$, and $M_{yz} = \iiint_Q x\delta(x,y,z)\,dV$

   The center of mass of L is $(\overline{x},\overline{y},\overline{z})$, where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$

   Definition: The **Moment of Inertia** (or the *angular mass*) of an object is a constant that reflects the inertia of a rigid object when it is revolved about some axis (or plane) of rotation. For a 2D Lamina, one can compute the moment of inertia when revolving about the x-axis (I_x), the y-axis (I_y), and about the origin (I_0). For a 3D solid, one can compute the moment of inertia when rotating about the x-axis (I_x), the y-axis (I_y), and about the z-axis (I_z).

3. **Second Moments or Moments of Inertia:**

   (a) 2D case: Suppose that L is 2-dimensional lamina in the shape of a region R in the plane and that $\delta(x,y)$ is a function that gives the density per unit area at each point of R. Then the moment of inertia of L with respect to the x and y axes and about the origin are:

   $I_x = \iint_R y^2\delta(x,y)\,dA$, $I_y = \iint_R x^2\delta(x,y)\,dA$, and $I_0 = \iint_R \left(x^2+y^2\right)\delta(x,y)\,dA$ [We'll often change this into polar coordinates.]

   (b) 3D case: Suppose that L is 3-dimensional solid in the shape of a solid region Q in space and that $\delta(x,y,z)$ is a function that gives the density per unit of volume at each point of Q. Then the moments of inertia of L with respect to the x, y, and z axes are:

   $I_x = \iiint_Q \left(y^2+z^2\right)\delta(x,y,z)\,dV$, $I_y = \iiint_Q \left(x^2+z^2\right)\delta(x,y,z)\,dV$, and $I_z = \iiint_Q \left(x^2+y^2\right)\delta(x,y,z)\,dV$