Math 102

Final Exam Practice Problems - Part 2

1. Compute the value of each of the following:

   (a) $\frac{7!}{4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} = 7\cdot6\cdot5 = 210$

   (c) $\frac{8!}{12!} = \frac{1}{12\cdot11\cdot10\cdot9} = \frac{1}{11{,}880}$

   (b) $P(12,7) = \frac{12!}{5!} = 12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6 = 3{,}991{,}680$

   (d) $C(10,7) = \frac{10!}{3!7!} = \frac{10\cdot9\cdot8}{3\cdot2\cdot1} = 120$

2. A company that has 750 employees wants to assign an copy code to each one of their employees. The copy machine has a number pad that only has the digits 1 through 5 on it. What is the smallest number of digits the codes can have is each employee is given their own unique copy code?

   Since there are 5 possible numbers to use for each digit in the code, we are looking at powers of 5 here. There are 5 codes 1 digit long, $5^2 = 25$ codes 2 digits long, $5^3 = 125$ codes 3 digits long, $5^4 = 625$ codes 4 digits long, and $5^5 = 3125$ codes 5 digits long. Therefore, the company needs to assign codes that are 5 digits long in order to have a unique code for all 750 of its employees.

3. Suppose a laundry basket contains 5 red socks and 7 white socks. How many ways can the socks be folded into pairs:

   (a) If we insist that each pair of socks has two socks of the same color.

   There are $C(5,2)$ ways to choose the first pair of red socks, and $C(3,2)$ ways to choose the second pair of red socks. There will be a single unmatched red sock left over after these have been chosen.

   Similarly, there are $C(7,2)$ ways to choose the first pair of white socks, $C(5,2)$ ways to choose the second pair of white socks, and $C(5,2)$ ways to choose the second pair of white socks. There will be a single unmatched white sock left over after these have been chosen.

   Combining these, there are $C(5,2)\cdot C(3,2)\cdot C(7,2)\cdot C(5,2)\cdot C(3,2) = 10\cdot3\cdot21\cdot10\cdot3 = 18{,}900$ ways to fold the socks into pairs of matching colors.

   (b) If we don't care what colors end up together in each pair.

   Here, we just think of matching up all 12 sock, without regard to the color of the socks. Then we have:

   $C(12,2)\cdot C(10,2)\cdot C(8,2)\cdot C(6,2)\cdot C(4,2)\cdot C(2,2) = 66\cdot45\cdot28\cdot15\cdot6\cdot1 = 7{,}484{,}400$ ways to fold the socks into pairs.

4. Suppose you and 5 friends are taking a summer road trip to California in a rented minivan that has 7 seats.

   (a) How many ways can the six of you be seated in the van?

   Note that someone must sit in the drivers seat, so we think of first picking a driver, then picking a seat to remain empty, and then, we arrange the remaining passengers in the 5 other occupied seats, so there are:

   $6\cdot6\cdot P(5,5) = 6\cdot6\cdot5! = 4{,}320$ possible seating arrangements that meet these conditions.

   (b) What if only three of you are over 21, and only people over 21 are legally allowed to drive the van?

   The counting is similar, except when we choose the driver, there are only 3 qualified people to choose from, so there are:

   $3\cdot6\cdot P(5,5) = 3\cdot6\cdot5! = 2{,}160$ possible seating arrangements that meet these conditions.

   (c) What if only 3 of you are over 21, and your friend Bob, who is under 21, refuses to sit anywhere else but in the front passenger seat?

   Here, we first pick a driver, then put Bob in his seat, then pick an empty seat from the 5 remaining seat, and finally, we arrange the 4 remaining passengers in the four remaining seats, so there are:

   $3\cdot1\cdot5\cdot P(4,4) = 3\cdot5\cdot4! = 360$ possible seating arrangements that meet these conditions.

5. Suppose you work at a job delivering pizzas. You are given 7 pizzas to deliver in West Fargo. How many different ways can you complete these deliveries?

   Since you can deliver the pizzas in any order you wish, and order does matter (at least to those waiting for their pizzas!), there are:

   $P(7,7) = 7! = 5{,}040$ ways to deliver the pizzas.

6. A bag contains 10 green balls, 5 red balls, and 4 orange balls.

   (a) Suppose 2 balls are drawn from the bag at the same time, without replacement.

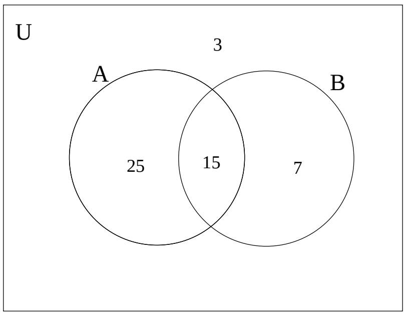
      i. Find the probability that both balls are green.
      Notice that there are 19 balls to choose from on the first draw, 10 of which are green, and there are 18 balls to choose from on the second draw, and if the first one drawn was green, 9 of the remaining balls are green. Then the probability of getting two green balls is:
      $\left(\frac{10}{19}\right)\left(\frac{9}{18}\right) = \frac{90}{342} = \frac{5}{19} \approx .263$, or 26.3%

      ii. Find the probability that both balls are the same color.
      Here, we use similar methods to figure out the probability that both balls are green, or both are red, or both are orange, which since these events are independent, we get by adding the probabilities or each of these individual events:
      $\left(\frac{10}{19}\right)\left(\frac{9}{18}\right) + \left(\frac{5}{19}\right)\left(\frac{4}{18}\right) + \left(\frac{4}{19}\right)\left(\frac{3}{18}\right) = \frac{90+20+12}{342} = \frac{122}{342} = \frac{61}{171} \approx .3567$, or 35.67%

      iii. Find the *odds* in favor of drawing one green and one red ball.
      Notice that there are $(5)(10) = 50$ ways of drawing Red/Green, and 50 ways of drawing Green/Red, which means 100 out of the 342 possible 2 ball combinations have 1 red and 1 green ball. Thus the odds in favor of drawing one green and one red are 100:242.

   (b) Rework the previous problem, but now assume that the two balls are drawn with replacement.
   Since the drawing is now done with replacement, the counting will be similar, but there will all 19 of the original balls will be available on each of the two draws, and there are $(19)(19) = 361$ ways to choose two balls if we replace between draws. Then the probabilities are:
   (i) Two green: $\left(\frac{10}{19}\right)\left(\frac{10}{19}\right) = \frac{100}{361} \approx .277$, or 27.7%
   (ii) Two the same color: $\left(\frac{10}{19}\right)\left(\frac{10}{19}\right) + \left(\frac{5}{19}\right)\left(\frac{5}{19}\right) + \left(\frac{4}{19}\right)\left(\frac{4}{19}\right) = \frac{100+25+16}{361} = \frac{141}{361} \approx .3906$, or 39.06%
   (iii) There are still 100 ways of drawing 1 red ball and 1 green ball, so the odds in this case are 100:261

7. A survey of 50 people finds that 40 of them like football, 22 of them like baseball, and 15 like both. Suppose that one participant is randomly selected from among all the people who participated in the survey.
   Note: To find these probabilities, a Venn Diagram is quite useful.



   (a) Find the probability that the person likes baseball but not football.
   From the diagram above, we see that the probability is: $\frac{7}{50} = .14$

   (b) Find the probability that the person likes either football or baseball.
   Again from the diagram above, we see that the probability is: $\frac{47}{50} = .94$

   (c) Find the probability that the student is likes neither football nor baseball.
   Here, the probability is: $\frac{3}{50} = .06$

   (d) Given that the person likes football, what is the probability that the person also likes baseball?
   Since 40 like football, of of these 40, 15 like baseball, the conditional probability is: $\frac{15}{40} = \frac{3}{8} = .375$

8. Consider the following game: A bag contains 8 red balls and 3 green balls. There are two ways to play -

   Option 1: Pay \$2 for the opportunity to draw one ball out of the bag. If you draw a red ball, you lose your \$2. If draw a green ball, you win \$5 (your original \$2, plus \$3 more).

   Option 2: Pay \$5 for the opportunity to first flip a coin, and then draw one ball out of the bag. If the you flip heads, then if you draw a red ball, you lose your \$5, while if you draw a green ball, you win \$10, your original \$5 plus \$5 more. If you flip tails, then if you draw a green ball, you lose your \$5, while if you draw a red ball, you win \$8, your original \$5 plus \$3 more.

   (a) Find the expected value for playing Option 1 of this game. Is this game fair?
   $\left(\frac{8}{11}\right)(-2)+\left(\frac{3}{11}\right)(+3)=\frac{-16}{11}+\frac{9}{11}=\frac{-7}{11}\approx -.636$, so we expect to lose 63.4 cents on average each time we play the game, so the game is not fair.

   (b) Find the expected value for playing Option 2 of this game. Is this game fair?

   We will compute this by finding the expected value of the heads case, and adding it to the expected value of the tails case:
   $\left[\left(\frac{1}{2}\right)\left(\frac{8}{11}\right)(-5)+\left(\frac{1}{2}\right)\left(\frac{3}{11}\right)(+5)\right]+\left[\left(\frac{1}{2}\right)\left(\frac{3}{11}\right)(-5)+\left(\frac{1}{2}\right)\left(\frac{8}{11}\right)(+3)\right]=\frac{-40+15-15+24}{22}=\frac{-16}{22}=-\frac{8}{11}\approx -.7272$, so we expect to lose 72.72 cents on average each time we play the game, so the game is not fair.

9. Suppose 2 cards are drawn without replacement from a deck of 52 cards. Find the probability that:

   (a) two cards from the same suit are drawn.
   $\left(\frac{52}{52}\right)\left(\frac{12}{51}\right)=\frac{12}{51}\approx .2353$, or 23.53%

   (b) two face cards are drawn.
   $\left(\frac{12}{52}\right)\left(\frac{11}{52}\right)=\frac{132}{2652}\approx .0498$, or 4.98%

   (c) a pair is drawn.
   $\left(\frac{52}{52}\right)\left(\frac{3}{51}\right)=\frac{1}{17}\approx .0588$, or 5.88%

   (d) a pair of aces is drawn.
   $\left(\frac{4}{52}\right)\left(\frac{3}{51}\right)=\frac{12}{2652}\approx .00452$, or 0.452%

10. A researcher wants to find out who will win the upcoming presidential election. To find out about this, she goes out to the local shooting range and asks people to fill out a survey. Although many people are not willing to take the time to answer, she eventually gets 50 responses. Of those that responded, 11 people say they plan to vote for Barack Obama, 7 people say they plan to vote for Hillary Clinton, and 32 people say they plan to vote for John McCain.

    (a) What is the population in this survey? What is the sample?

    Population: people in the U.S.

    Sample: the people who took the survey (the 50 respondents, perhaps plus those who refused to answer)

    (b) What forms of bias, if any, may have effected the data collected in this survey? Explain your reasoning.

    There is sample bias, since they did not choose a random sample. Only those people who happened to be at the local shooting range on one particular day had a chance to participate in the survey and this is almost certainly not representative of people in the U.S as a whole.

    There is also non-response bias, since several people chose not to answer the survey question.

    (c) Based on this study, what conclusions, if any, can be reached about who will win the upcoming Presidential election? Explain your reasoning.

    The design of this study is significantly flawed. The sample is especially poor, since people at a shooting range are likely to have strong views on hunting and gun control and so would be more likely to vote for a candidate who shares their views on these issues. There is no way we could generalize the information from this biased sample to the entire state let alone to other parts of the country. No real conclusions can be reached based on this study.

11. (a) Give an example of a real life situation where the mean is the most appropriate measure of central tendency.

    Computing one students average score on 4 midterm exams given during a single semester.

    (b) Give an example of a real life situation where the median is the most appropriate measure of central tendency.

    Computing the average household income in the United States. (The median would be better than the mean because there are many positive outliers, like Bill Gates)

(c) Give an example of a real life situation where the mode is the most appropriate measure of central tendency.
Computing the most popular fact food restaurant chain. (how you you compute the median of mean of Taco Bell and McDonalds?)

12. Suppose that you got scores of 78, 85, 82, and 75 on 4 exams worth 100 points each.

(a) What grade would you need to get on a 200 point final exam in order to end up with an average of 70 percent?
Since you have earned a total of $78 + 85 + 82 + 75 = 320$ points already, and there will be 600 points altogether, to get a 70% average, you would need $(.7)(600) = 420$ total points, so you would need to score 100 out of 200 on the final exam.

(b) What grade would you need to get on a 200 point final exam in order to end up with an average of 80 percent?
Here, you would need $(.8)(600) = 480$ total points, so you would need to score 160 out of 200 on the final exam.

(c) What grade would you need to get on a 200 point final exam in order to end up with an average of 90 percent?
You would need $(.9)(600) = 540$ total points, so you would need to score 220 out of 200 on the final exam, which is impossible, of course.

13. Given the data set $\{2, 5, 12, 21, 27, 13, 6, 11, 5, 12, 24, 7\}$

(a) Find the mean, median, and mode of this data set.
Rearranging the data, we have: $\{2, 5, 5, 6, 7, 11, 12, 12, 13, 21, 24, 27\}$
Then the mean is: $\overline{x} = \frac{2+5+5+6+7+11+12+12+13+21+24+27}{12} = \frac{145}{12} \approx 12.08$
The median is: $\frac{11+12}{2} = 11.5$
There are actually two modes: 5 and 12, since both occur twice while all other data values occur only once.

(b) Make a stem and leaf display for this data set.

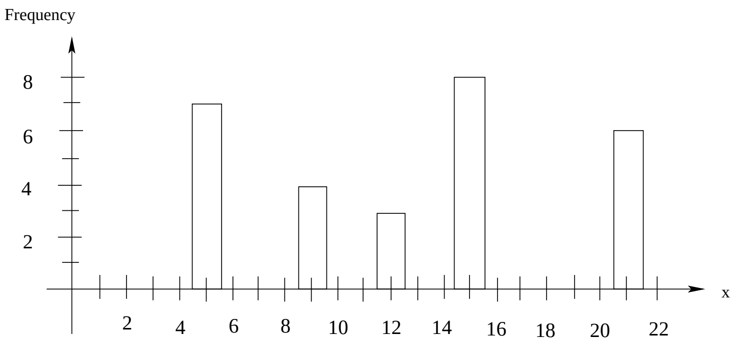
0 | 2 5 5 6 7
1 | 1 2 2 3
2 | 1 4 7

(c) Find the 5 number summary of this data set.
min: 2
max: 27
median: 11.5
$Q_1 : \frac{5+6}{2} = 5.5$
$Q_3 : \frac{13+21}{2} = 17$

14. Given the following frequency table:

x	frequency
5	7
9	4
12	3
15	8
21	6

Frequency Histogram:



(a) In the space provided above, make a frequency histogram for the data in the table above.

(b) Compute the mean and median of the data in this table.
mean: $\overline{x} = \frac{(5)(7)+(9)(4)+(12)(3)+(15)(8)+(21)(6)}{28} = \frac{353}{28} \approx 12.607$
median: $\frac{12+15}{2} = 13.5$

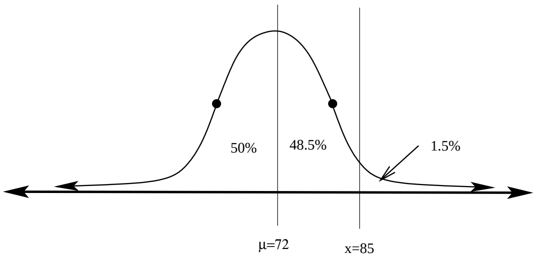
15. Find the mean and standard deviation of the data set: $\{3, 7, 12, 14, 17, 25\}$

x	$x - \overline{x}$	$(x - \overline{x})^2$
3	$3 - 13 = -10$	100
7	$7 - 13 = -6$	36
12	$12 - 13 = -1$	1
14	$14 - 13 = 1$	1
17	$17 - 13 = 4$	16
25	$25 - 13 = 12$	144

First notice that $\overline{x} = \frac{3+7+12+14+17+25}{5} = \frac{78}{6} = 13$.
Also, $n = 6$, so $n - 1 = 5$
By adding the entries in the last column of table on the left, we see that $\Sigma(x - \overline{x})^2 = 298$.
Then $s = \sqrt{\dfrac{\Sigma(x-\overline{x})^2}{n-1}} = \sqrt{\dfrac{298}{5}} \approx 7.72$

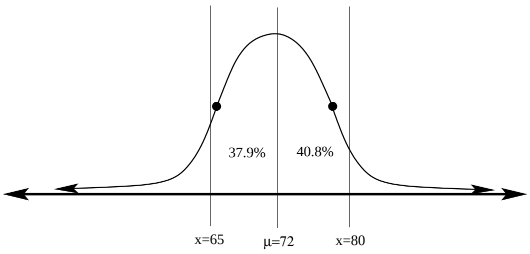
16. Suppose that 500 test scores (on a 100 point test) are approximately normally distributed with a mean of 72, and a standard deviation of 6.

(a) What percentage of scores are above 85 points?
$z = \frac{85-72}{6} = 2.17$. From the z-table, we look up this entry and find the corresponding area to be $A = .485$, or 48.5%. This represents the area under the normal curve between $x = 72$ and $x = 85$. To find the percentage of scores *above* 85, we must subtract: $50 - 48.5\% = 1.5\%$.



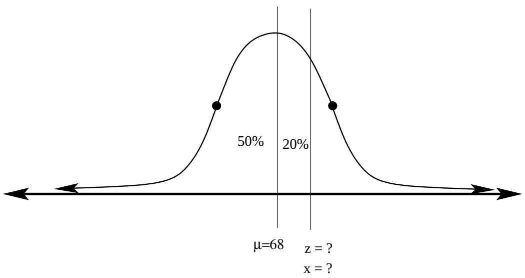
(b) How **many** scores are below 50 points?
First notice that the z-score for $x = 50$ is $z = \frac{50-72}{6} = -3.67$. From the z-table, we look up this entry and find that there is no corresponding entry. This is due to the fact that this z entry is so large that the area between is and the mean is nearly 50% Because of this, we conclude that there are no scores below 50 points on this exam.

(c) What percentage of scores are between 65 and 80 points?
Here, we need to compute the z-score for each of these x-values:
$z_1 = \frac{80-72}{6} = 1.33$, and $z_2 = \frac{65-72}{6} = -1.17$.
Next, we look up the area values for each of these in the z-table: $A_1 = .408$ and $A_2 = .379$



From the figure above, we see that to find the percentage of scores between the two, we add the areas we found above: $.408 + .379 = .787$, or 78.7%.

(d) What score would a person need to get on the test in order to have scored higher than 70% of the people who took this test?



We need to find the x-value which is above 70% of all scores. To do this, we first need to find the appropriate z-score. We look in the table for the z-score whose area correxponds to 20%, and find that $A = .200$ when $z = .525$. Working backwards from this, $x = z \cdot \sigma + \mu = (.525)(6) + 72 = 75.15$.

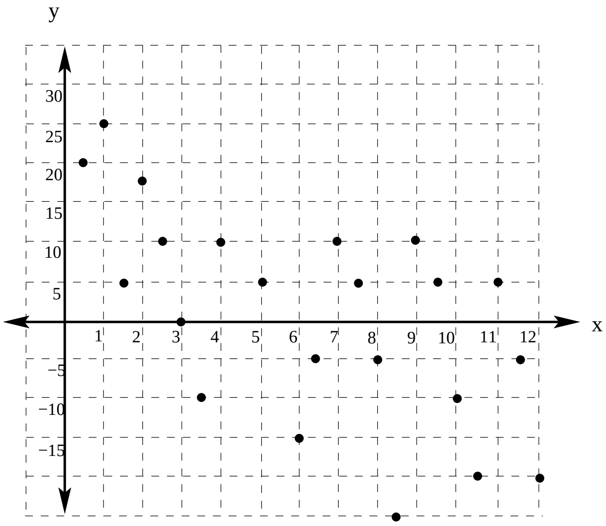
17. A 16oz jar of peanut butter cost \$1.78 in 1995. In 2005, a similar jar cost \$2.99.

   (a) Find a line that models the price of peanut butter over time (hint: you can take $x = 0$ to represent 1995)
   Using the points $(0, 1.78)$ and $(10, 2.99)$, we find $m = \frac{2.99-1.78}{10-0} = .121$ and $b = 1.78$.
   Therefore, the line modeling the price of peanut butter is given by: $y = .121x + 1.78$, where $x = 0$ corresponds to the year 1995.

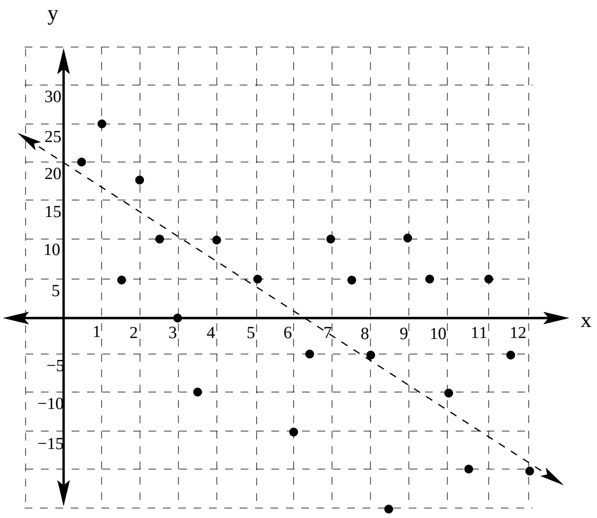
   (b) Use your model to predict the price of peanut butter in 2010.
   2010 corresponds to $x = 2010 - 1995 = 15$, and so $y = .121(15) + 1.78 = \$3.595$, or around \$3.60.

   (c) According to your model, when will the price of peanut butter reach \$5.00 for a 16oz jar?
   If $y = \$5.00$, then $5 = .121x + 1.78$, so $5 - 1.78 = .121x$, or $3.22 = .121x$
   Therefore, $x = \frac{3.22}{.121} = 26.61$.
   Hence, according to this model, the price of peanut butter will reach \$5 per 16 oz jar 26.61 years after 1995, or sometime during 2022.

18. Given the scatter plot shown here:



(a) Sketch in an estimated least squares regression line for this data.



(b) Is the correlation coefficient positive or negative for this data?

Since the slope of the least squares regression line is negative, the correlation coefficient will be negative as well. This is due to the fact that in general, as x increases, y decreases, so these two variables are negatively correlated.

(c) is the correlation coefficient closer to 0, 1, or -1?

Although the fit of the least squares regression line is far from perfect, it is not too bad a fit. The data shows a fairly string negative correlation, so that correlation coefficient is probably closer to -1 than to 0 or $+1$.