Math 102
Exam 2: Additional Practice Problems

1. Compute the value of each of the following:

   (a) $0! = 1$

   (b) $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

   (c) $P(12, 7) = \frac{12!}{5!}$
   $= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 3,991,680$

   (d) $\frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720$

   (e) $C(10, 7) = \frac{10 \cdot 9 \cdot 8}{3!} = 120$

   (f) $\frac{1000!}{997!} = 1000 \cdot 999 \cdot 998$
   $= 997,002,000$

2. A local restaurant has 4 appetizers, 12 entrees, and 5 desserts. Find the number of possible meals that can be formed by choosing exactly one menu item of each type.

   Using the fundamental counting principle, there are $(4)(12)(5) = 240$ possible meals.

3. A company assigns billing codes to each of its clients consisting of two letters followed by three one-digit numbers. Find the number of possible billing codes if:

   (a) Repetition is allowed.

   There are $(26)(26)(10)(10)(10) = 676,000$ possible billing codes if repetition is allowed.

   (b) Repetition is **not** allowed.

   There are $(26)(25)(10)(9)(8) = 468,000$ possible billing codes if repetition is not allowed.

4. A club has 5 male members and 7 female members.

   (a) How many ways can a committee of 5 club members be chosen?

   Since there are 12 total members in the club, and no restrictions on who is on the committee, and the order people are put onto the committee does not matter, the number of possible ways of forming this committee is given by: $C(12, 5) = \frac{12!}{7!5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792$

   (b) How many ways can a committee of 5 be chosen if the committee is required to consist of 2 men and 3 women?

   Since we must have exactly 2 men and 3 women on the committee, and the order people are put onto the committee does not matter, the number of possible ways of forming this committee is given by: $C(5, 2) \cdot C(7, 3) = \frac{5!}{3!2!} \frac{7!}{4!3!} = \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$

   (c) How many ways can a committee of 5 be chosen if one member is designated as the head of the committee, and the rest of the committee is required to consist of 2 men and 2 women?

   There are two ways to count this one. First, we can think about choosing the head of the committee first, in which case, we will need to consider two cases, the case when the head is male, and the case when the head is female. Since there are 5 men and 7 women in the club to choose from, and we want two of each to be on the committee in addition to the head, we have:
   $5 \cdot C(4, 2) \cdot C(7, 2) + 7 \cdot C(5, 2) \cdot C(6, 2) = 5 \cdot \frac{4!}{2!2!} \frac{7!}{5!2!} + 7 \cdot \frac{5!}{3!2!} \frac{6!}{4!2!} = 5 \cdot \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{7 \cdot 6}{2 \cdot 1} + 7 \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{6 \cdot 5}{2 \cdot 1} = 630 + 1050 = 1680$

   A slightly easier way to count this is to think of choosing the 2 male and 2 female committee members first, and then choosing the head from the remaining 8 club members. This gives: $C(5, 2) \cdot C(7, 2) \cdot 8 = \frac{5!}{3!2!} \frac{7!}{5!2!} \cdot 8 = \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{7 \cdot 6}{2 \cdot 1} \cdot 8 = 1680$

5. Suppose you go to the store and purchase a variety pack with 8 individually wrapped bags of chips, where each bag is of a different type. You plan to select one bag each day (Monday through Friday) to include as part of the lunch you take to work. How many different ways could the chips you bring to work for lunch that week be selected?

   Since we are bringing different bags of chips on different days, order matters, so this is a permutation problem. Therefore the count is given by: $P(8, 5) = \frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$

6. A bag contains 7 white chips, 3 red chips, and 2 blue chips. Suppose two chips are drawn from the bag.

   (a) How many ways could you draw 2 red chips?

   Since we only care about which chips you end up with, order does not matter, so we will use combination counting. There are 3 red chips, and we want to draw two of them, so there are $C(3,2) = \frac{3!}{1!2!} = 3$ ways of drawing two red chips.

   (b) How many ways could you draw 2 chips that are both the same color?

   We will once again use combination counting. However, notice that there are three different types of outcomes that result in chips of the same color: 2 red, 2 white, and 2 blue.
   There are $C(3,2) = \frac{3!}{1!2!} = 3$ ways of drawing two red chips.
   There are $C(7,2) = \frac{7!}{5!2!} = \frac{7 \cdot 6}{2} = 21$ ways of drawing two white chips.
   There are $C(2,2) = \frac{2!}{1!2!} = 1$ way of drawing two blue chips.
   Therefore, there are $3 + 21 + 1 = 25$ ways of drawing two chips of the same color. Notice that we add rather than multiply these combinations because they represent separate ways of getting two chips of the same color and are not steps within the same counting procedure.

   (c) How many ways could you draw two chips that are **not** red?

   Since there are 12 chips total, and 3 of them are red, 9 of the chips are not red.
   Therefore, there are $C(9,2) = \frac{9!}{7!2!} = \frac{9 \cdot 8}{2} = 36$ ways of drawing two chips that are both not red.

   (d) How many ways could you draw **exactly one** blue chip?

   Notice that there are 2 blue chips and 10 non-blue chips. Then there are $2 \cdot 10 = 20$ ways of draiwing one blue chip and one non-blue chip (or exactly one blue chip).

7. For a standard deck of 52 cards, find the number of 5 card hands satisfying each description.

   (a) All 5 of the cards are spades.

   Since we are counting 5 card poker hands, we do not care what order the cards are dealt, only which cards end up in the hand, so we will be using combination counting.
   With this in mind, since there are 13 spade cards, there are $C(13,5) = \frac{13!}{5!8!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!} = \frac{154{,}440}{120} = 1287$ 5 card hands consisting of all spades.

   (b) Four of the cards are red and one of the cards is black.

   Since there are 26 red cards and 26 black cards, there are $C(26,4) \cdot C(26,1) = \frac{26!}{22!4!} \cdot 26 = \frac{26 \cdot 25 \cdot 24 \cdot 23}{4!} \cdot 26 = \frac{358{,}800}{24} \cdot 26 = 14{,}950 \cdot 26 = 388{,}700$ 5 card hands consisting of four red cards and one black card.

   (c) Three of the cards are hearts and two of the cards are clubs.

   Since there are 13 hearts and 13 clubs, there are $C(13,3) \cdot C(13,2) = \frac{13!}{10!3!} \cdot \frac{13!}{11!2!} = \frac{13 \cdot 12 \cdot 11}{6} \cdot \frac{13 \cdot 12}{2} = (286)(78) = 22{,}308$ 5 card hands consisting of 3 hearts and 2 clubs.

   (d) There is an Ace, two 7s, and two face cards.

   Recall that since Jacks, Queens, and Kings are face cards, there are exactly 12 face cards in a standard deck.
   Then, there are 4 ways of choosing an Ace, there are $C(4,2) = 6$ ways of choosing two 7s, and there are $C(12,2) = \frac{12!}{10!2!} = \frac{12 \cdot 11}{2} = 66$ ways of choosing two face cards.
   Therefore there are $4 \cdot 6 \cdot 66 = 1584$ ways of getting a hand with an Ace, two 7s, and two face cards.

8. (a) Use set notation to list all the elements of the set:
   $A = \{\ x \mid x$ is an odd whole number less than 21 $\}$
   $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\ \}$

   (b) Use set-builder notation to describe the set $\{-3, -2, -1, 0, 1, 2, 3\}$
   $\{\ x \mid x$ is an integer that is greater than -4 and less than 4 $\}$

9. Given that $A = \{ x \mid x \text{ is a letter in the word } banana \}$, $B = \{ x \mid x \text{ is a letter in the word } bandana \}$, $C = \{ x \mid x \text{ is a letter in the word } band \}$, and $D = \{ \emptyset \}$, indicate whether the following are True or False (you do NOT need to justify your answers)

(a) $\{a\} \in A$ False: $\{a\}$ is a set not an element

(b) $d \in A$ False

(c) $\emptyset \subseteq A$ True: $\emptyset$ is a subset of **any** set.

(d) $A \subseteq B$ True

(e) $B \subset C$ False

(f) $B = C$ True.

(g) $D \subset B$ False: $\emptyset$ is not an element of B

(h) $\emptyset \in D$ True.

10. Let $U = \{ x \mid x \text{ is a positive integer less than } 12 \}$; $A = \{0, 2, 4, 6, 8\}$; $B = \{1, 2, 3, 4, 5\}$; $C = \{6, 7, 8, 9, 10\}$; $D = \{ x \mid x \text{ is an element of both } A \text{ and } C\}$

(a) Write U in roster notation.

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

(b) Find $n(A)$ and $n(D)$.

First, $n(A) = 5$. Also, notice that $D = \{6, 8\}$, so $n(D) = 2$.

(c) Which of the sets A, B, C and D are equivalent to each other?

Since A, B, and C all have 5 elements, they are all equivalent to each other. D is not equivalent to any of the others since $n(D) = 2$.

(d) Which of the sets A, B, C and D are equal?

None of these sets are equal to each other since none of them share precisely the same elements.

(e) Is A a subset of U? Justify your answer.

No. Notice that $0 \in A$ but $0 \notin U$.

11. (a) List all the subsets of the set $\{a, b, c\}$

$\emptyset$
$\{a\}$; $\{b\}$; $\{c\}$
$\{a, b\}$; $\{a, c\}$; $\{b, c\}$
$\{a, b, c\}$

(b) How many subsets does the set $\{a, b, c, d, e\}$ have?
Since $\{a, b, c, d, e\}$ has 5 elements, it has $2^5 = 32$ different subsets

(c) Write out the first 6 rows of Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(d) Use Pascal's Triangle to find the number of 3 element subsets of $\{a, b, c, d, e\}$
Since $\{a, b, c, d, e\}$ has 5 elements in it, we look at the 5th row in the triangle above. Since the 4th entry in this row corresponds to the number of subsets of size 3, there are 10 three element subsets of the set $\{a, b, c, d, e\}$