Math 487
Chapter 4
Practice Problem Solutions

1. Give the definition of each of the following terms:

   (a) A complete quadrangle

   A **complete quadrangle** is a set of four points, no three of which are collinear, and the six lines incident with each pair of these points. The four points are called *vertices* and the six lines are called *sides* of the quadrangle.

   (b) A complete quadrilateral

   A **complete quadrilateral** is a set of four lines, no three of which are concurrent, and the six points incident with each pair of these lines. The four lines are called *sides* and the six points are called *vertices* of the quadrilateral.

   (c) A perspectivity between pencils of points

   A one-to-one mapping between two pencils of points is called a **perspectivity** if the lines incident with the corresponding points of the two pencils are concurrent. The point where the lines intersect is called the *center of the perspectivity.*

   (d) A perspectivity between pencils of lines

   A one-to-one mapping between two pencils of lines is called a **perspectivity** if the points of intersection of the corresponding lines of the two pencils are collinear. The line containing the points of intersection is called the *axis of the perspectivity.*

   (e) A projectivity between pencils of points

   A one-to-one mapping between two pencils of points is called a **projectivity** if the mapping is a composition of finitely many elementary correspondences or perspectivities.

   (f) The harmonic conjugate of a point C with respect to points A and B.

   Four collinear points A, B, C, D form a harmonic set, denoted $H(AB, CD)$, if A and B are diagonal points of a quadrangle and C and D are on the sides determined by the third diagonal point. The point C is the **harmonic conjugate** of D with respect to A and B.

   (g) A point conic

   A **point conic** is the set of points of intersection of corresponding lines of two projectively, but not perspectively, related pencils of lines with distinct centers.

   (h) A line conic

   A **line conic** is the set of lines that join corresponding points of two projectively, but not perspectively, related pencils of points with distinct axes.

2. State each of the following:

   (a) Desargues' Theorem

   If two triangles are perspective from a point, then they are also perspective from a line.

   (b) The Fundamental Theorem of Projective Geometry

   A projectivity between two pencils of points is uniquely determined by three pairs of corresponding points.

3. True or False

(a) In a plane projective geometry, if two triangles are perspective from a point, then they are also projective from a line.

True. This is a consequence of Desargues' Theorem

(b) In the Poincaré Half Plane, if two triangles are perspective from a point, then they are also projective from a line.

False. See Homework Exercise #4.18 [Hint: pick a pair of triangles with a pair of corresponding sides that are parallel.]

(c) In a plane projective geometry, if two triangles are perspective from a line, then they are also projective from a point.

True. This is a consequence of the dual of Desargues' Theorem.

(d) Every point in a plane projective geometry is incident with at least 4 distinct lines.

True. This is a consequence of the dual of Theorem 4.4, which is true since Plane Projective Geometries satisfy the principle of duality.

(e) If $H(AB, CD)$ then $H(CD, BA)$.

True. This is a consequence of Theorem 4.8.

(f) If $H(AB, CD)$ and $H(AB, C'D)$ then $C = C'$

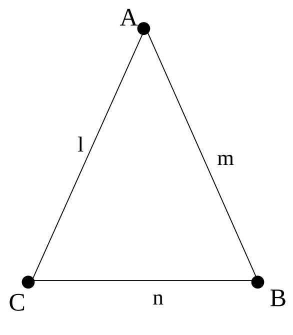
True. This is a consequence of the Fundamental Theorem 4.7.

(g) If A, B, C and A', B', C' are distinct elements in pencils of points with distinct axes p and p', there there exists a perspectivity such that $ABC \overset{o}{\doublebarwedge} A'B'C'$

False. Theorem 4.10 guarantees that there is a **projectivity** such that $ABC \wedge A'B'C'$, but this projectivity is not necessarily a perspectivity (for example, the construction we did in class to prove this theorem required two perspectivities).

4. Prove that Axiom 3 in independent of Axiom 1 and Axiom 2.

Consider the following model:

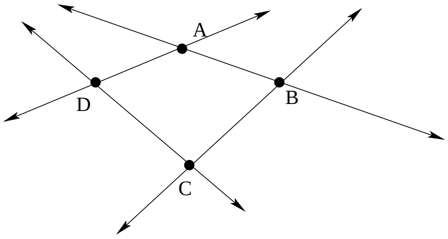


In this model, $A, B,$ and C are points, and $l, m,$ and n are lines. Notice that any pair of distinct points are on exactly one line [A and B are on m, A and C are on l, and B and C are on n]. Also notice that any two distinct lines are incident with at least one point [in fact, $l \cot m = A$, $l \cot n = C$, and $m \cot n = B$]. However, since there are only 3 points in this model, Axiom 3 is not satisfied.

5. (a) State and prove the dual of Axiom 3.

Recall Axiom 3 states: There exist at least four points, no three of which are collinear.
Then the Dual of Axiom 3 is: There exist at least four lines, no three of which are concurrent.

Proof: Let A, B, C, and D be four distinct points, no three of which are collinear (we know these points exist by Axiom 3). Using Axiom 1, the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{CD}$ all exist. Since no three of the points A, B, C, and D are collinear, these six lines must be distinct.



Consider the four lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$. To show that no three of these lines are concurrent, we proceed by contradiction. Suppose not. Then three of these lines would be concurrent. For example, suppose that $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CD}$ are concurrent. Using the Dual of Axiom 1, B is the only point of intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$. Therefore, B must be the point of concurrency for the three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CD}$. But then B is on $\overleftrightarrow{CD}$. This contradicts our assumption that B, C, and D are noncollinear. The other cases are similar.
Therefore, there exist at least four lines, no three of which are concurrent. □.

(b) State and prove the dual of Axiom 4.

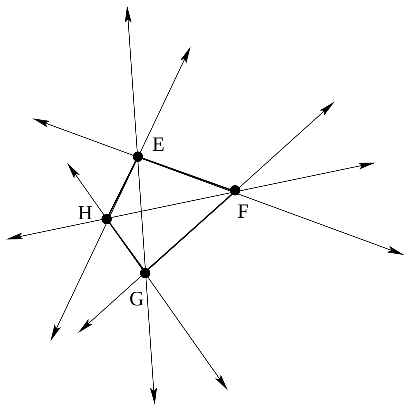
Recall that Axiom 4 states: The three diagonal points of a complete quadrangle are never collinear.
Then the Dual of Axiom 4 is: The three diagonal lines of a complete quadrilateral are never concurrent.

Proof: Let $abcd$ be a complete quadrilateral (we know that such a quadrilateral exists from the Dual of Axiom 3). Let $E = a \cdot b$, $F = b \cdot c$, $G = c \cdot d$, $H = a \cdot d$, $I = a \cdot c$ and $J = b \cdot d$. These points exist by Axiom 2, and are unique by the Dual of Axiom 1. Using Axiom 1, the diagonal lines $\overleftrightarrow{EG}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{IJ}$ exist.
Claim: The diagonal lines $\overleftrightarrow{EG}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{IJ}$ are not concurrent. We will prove this claim using proof by contradiction. Suppose that the lines $\overleftrightarrow{EG}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{IE}$ are concurrent. Then $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$ must be the point of concurrency between these lines. Therefore, the points I, J, and $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$ are collinear.
Since $abcd$ is a complete quadrilateral, no three of the lines $a = \overleftrightarrow{EH}$, $b = \overleftrightarrow{EF}$, $c = \overleftrightarrow{FG}$, and $d = \overleftrightarrow{GH}$ are concurrent. Thus, (using the dual of the argument in the proof of the Dual of Axiom 3) E, F, G, and H are four points, no three of which are collinear. Hence, $EFGH$ is a complete quadrangle with diagonal points $\overleftrightarrow{EF} \cdot \overleftrightarrow{GH} = b \cdot d = J$, $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$, and $\overleftrightarrow{EH}\overleftrightarrow{FG} = a \cdot c = I$. Hence, using Axiom 4, then the points I, J, and $\overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$ are noncollinear, which contradicts our previous assumption that they are collinear. Therefore, the diagonal lines of the complete quadrilateral $abcd$ are not concurrent. □.

6. (a) Prove that a complete quadrangle exists.

Proof: By Axiom 3, there are 4 distinct points no three of which are collinear. Call these points A, B, C, and D. By Axiom 1, the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{CD}$ all exist. We claim that these six lines are all distinct. To see this, first suppose that $\overleftrightarrow{AB} = \overleftrightarrow{AC}$. This would cause A, B, and C to be collinear, which contradicts our earlier assumption. The other cases are similar (note that in the case where we assume $\overleftrightarrow{AB} = \overleftrightarrow{CD}$ we have that A, B, C, and D are all collinear.)
Consequently, a complete quadrangle exists. □.

(b) Draw a model for a complete quadrangle $EFGH$.

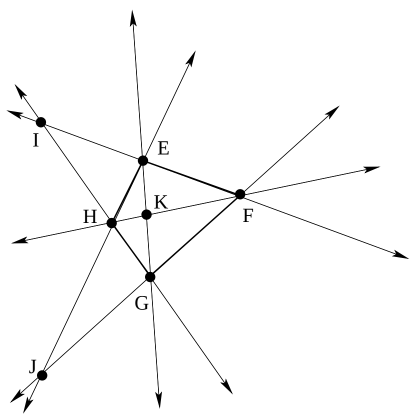


The points E, F, G, and H, along with the lines $\overleftrightarrow{EF}$, $\overleftrightarrow{EG}$, $\overleftrightarrow{EH}$, $\overleftrightarrow{FG}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{GH}$ form a complete quadrangle.

(c) Identify the pairs of opposite sides in the quadrangle $EFGH$.

There are 3 pairs of opposite sides in the quadrangle:
$\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$
$\overleftrightarrow{EH}$ and $\overleftrightarrow{FG}$
$\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$

(d) Construct and identify the diagonal points of the quadrangle $EFGH$.

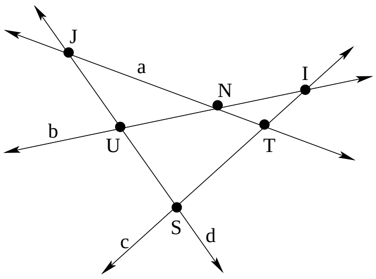


Let $I = \overleftrightarrow{EF} \cdot \overleftrightarrow{GH}$
Let $J = \overleftrightarrow{EH} \cdot \overleftrightarrow{FG}$
Let $K = \overleftrightarrow{EG} \cdot \overleftrightarrow{FH}$
Then I, J and K are the diagonal points of this complete quadrangle.

7. (a) Prove that a complete quadrilateral exists.

Proof: By Axiom 3, there are 4 distinct points no three of which are collinear. Call these points A, B, C, and D. By Axiom 1, the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{CD}$ all exist. As in the proof of the existence of a complete quadrangle, these six lines are all distinct, otherwise, three of the original points would be collinear contrary to our previous assumption.
Consider the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$. Using the dual of Axiom 1, let $E = \overleftrightarrow{AB} \cdot \overleftrightarrow{CD}$ and let $F = \overleftrightarrow{AD} \cdot \overleftrightarrow{BC}$. Notice that E and F must be distinct from A, B, C, and D, otherwise this would once again force 3 of our original points to be collinear, contrary to our previous assumption. From this, we see that no three of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$ are concurrent.
Hence the points A, B, C, D, E, and F along with the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$ form a complete quadrilateral. $\square$.

(b) Draw a model for a complete quadrilateral *abcd*.

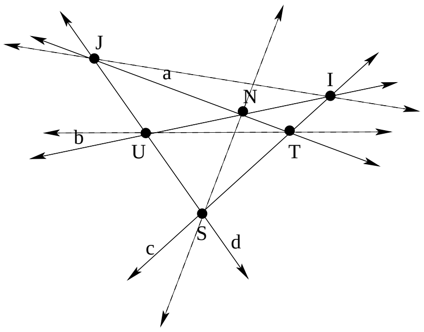


The lines a, b, c, and d along with the points J, U, S, T, I, and N form a complete quadrilateral.

(c) Identify the pairs of opposite points in the quadrilateral *abcd*.

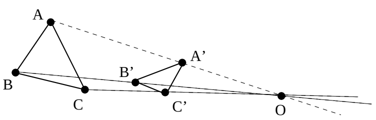
There are three pairs of opposite points in this quadrilateral:
J and I; U and T; S and N.

(d) Construct and identify the diagonal lines of the quadrilateral *abcd*.



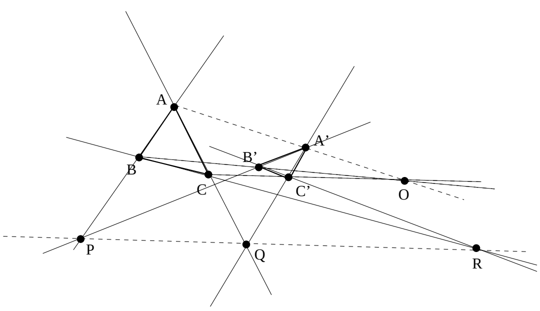
The diagonal lines in this quadrilateral are $\overleftrightarrow{JI}$, $\overleftrightarrow{UT}$, and $\overleftrightarrow{SN}$.

8. (a) Construct an example of two triangles that are perspective from a point. Be sure to identify the point O that the triangles are perspective from.



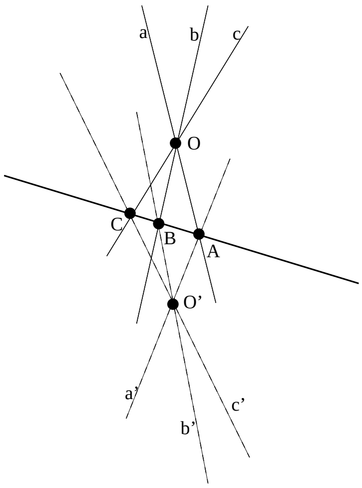
In the diagram above, $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the point O.

(b) Are these two triangles also perspective from a line? If so, identify the line that the triangles are perspective from. If not, explain why they cannot be perspective from a line.



From the diagram above, if we let $\overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'} = P$, $\overleftrightarrow{AC} \cdot \overleftrightarrow{A'C'} = Q$, and $\overleftrightarrow{BC} \cdot \overleftrightarrow{B'C'} = R$, notice that R is incident with the line $\overleftrightarrow{PQ}$, so $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the line $\overleftrightarrow{PQ}$.

9. Illustrate a projectivity from a pencil of lines a, b, c with center O to a pencil of lines a', b', c' with center $O' \neq O$.

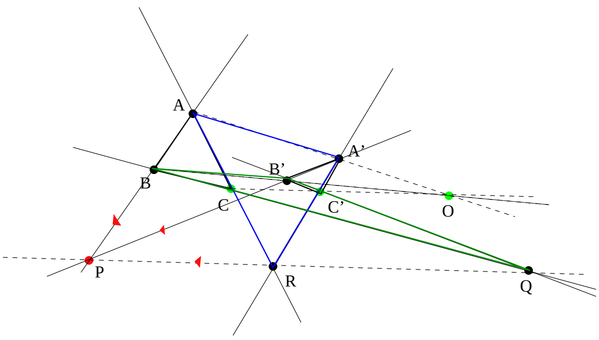


10. Prove each of the following:

(a) The dual of Desargues' Theorem

Dual of Desargues' Theorem: If two triangles are perspective from a line, then they are also perspective from a point.

Proof: Suppose $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line. Let $P = \overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'}$, $Q = \overleftrightarrow{BC} \cdot \overleftrightarrow{B'C'}$ and $R = \overleftrightarrow{AC} \cdot \overleftrightarrow{A'C'}$. By the definition of perspectivity from a line, the points P, Q and R are collinear. Let $O = \overleftrightarrow{AA'} \cdot \overleftrightarrow{BB'}$. To show that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent, we must show that O is on the line $\overleftrightarrow{CC'}$.
Consider the triangles $\triangle RAA'$ and $\triangle QBB'$. Since P, Q, R are collinear, P is on line $\overleftrightarrow{QR}$. Since $P = \overleftrightarrow{AB} \cdot \overleftrightarrow{A'B'}$, P is on line $\overleftrightarrow{AB}$ and line $\overleftrightarrow{A'B'}$. Hence triangles $\triangle RAA'$ and $\triangle QBB'$ are perspective from point P, by the definition of perspective from a point.
Hence by Axiom 5 (Desargues' Theorem), triangles $\triangle RAA'$ and $\triangle QBB'$ are perspective from a line. By definition of perspectivity from a line, the points $C = \overleftrightarrow{RA} \cdot \overleftrightarrow{QB}$, $C' = \overleftrightarrow{RA'} \cdot \overleftrightarrow{QB'}$ and $O = \overleftrightarrow{AA'} \cdot \overleftrightarrow{BB'}$ are collinear. Hence O is on the line $\overleftrightarrow{CC'}$. Therefore $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent. Therefore, $\triangle ABC$ and $\triangle A'B'C'$ are perspective from point O. $\square$.



(b) Theorem 4.6

Theorem: If A, B, and C are three distinct collinear points, then a harmonic conjugate of C with respect to A and B exists.

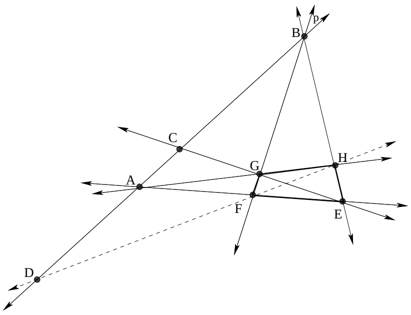
Proof: Let A, B, and C be three distinct collinear points. By Axiom 3, there is a point E such that A, C and E are non-collinear. By Theorem 4.3, there is a point F on $\overleftrightarrow{AE}$ that is distinct from A and E. Let $G = \overleftrightarrow{CE} \cdot \overleftrightarrow{BF}$ and let $H = \overleftrightarrow{AG} \cdot \overleftrightarrow{BE}$.
Claim: The points E, F, G, and H and the lines $\overleftrightarrow{EF}$, $\overleftrightarrow{EG}$, $\overleftrightarrow{EH}$, $\overleftrightarrow{FG}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{GH}$ determine a complete quadrangle.

To see this, notice that the points E, F, G and H are distinct. E and F are distinct by construction. For the others, first suppose that $G = F$. Since A is incident to $\overleftrightarrow{EF}$ and $G = F$ is incident to $\overleftrightarrow{CE}$, then $A, E, G = F, C$ is a collinear set, contrary to our previous assumptions. The other cases are similar.

Next, Suppose that E, F and G are collinear. Since G is incident to $\overleftrightarrow{BF}$, F is incident to $\overleftrightarrow{AE}$, and A is incident to $\overleftrightarrow{AB}$, then A, C, and E are collinear, contrary to our previous assumptions. The other cases are similar.

This proves the claim.

Notice that $\overleftrightarrow{FH}$ is the remaining side of the complete quadrangle. Then if we take $D = \overleftrightarrow{FH} \cdot \overleftrightarrow{AB}$, then we have constructed the harmonic set $H(AB, CD)$. $\square$.



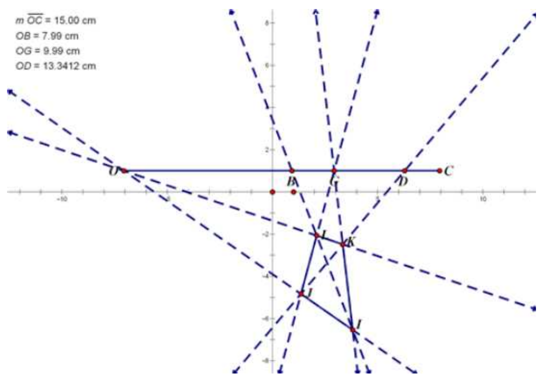
(c) The Fundamental Theorem of Projective Geometry

Theorem: A projectivity between two pencils of points is uniquely determined by three pairs of corresponding points.

Proof: We must show that if A, B, C, and D are in a pencil of points with axis p and A', B', C' are in a pencil of points with axis p', then there exists a unique point D' on p' such that $ABCD \wedge A'B'C'D'$.

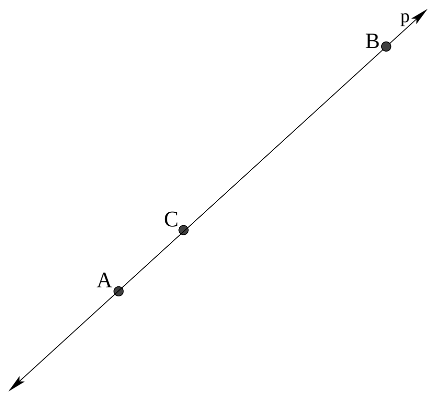
Assume A, B, C, and D are in a pencil of points with axis p and that A', B', and C' are in a pencil of points with axis p'. Recell that there exists a point D' on p' such that $ABCD \wedge A'B'C'D'$ (to find D, we find d the image of D under the first elementary correspondance, and then find the image of d under the second elementary corespondance, and continue through each of the finitely many elementary correspondances in the projectivity). Suppose there is a projectivity and a point D'' such that $ABCD \wedge A'B'C'D''$. Since $A'B'C'D' \wedge ABCD$ and $ABCD \wedge A'B'C'D''$, we have $A'B'C'D' \wedge A'B'C'D''$. Therefore, using Axiom 6, $D' = D''$. $\square$.

11. The frequency ratio $3:4:5$ is also equivalent to the ratio $\frac{3}{2} : \frac{15}{8} : \frac{9}{8}$, which gives the chord G, B, D called the dominant of the major triad of the example above. Show $H(OG, DB)$ where $OG = (\frac{2}{3})OC$, $OB = (\frac{8}{15})OC$, and $OD = (\frac{8}{9})OC$.



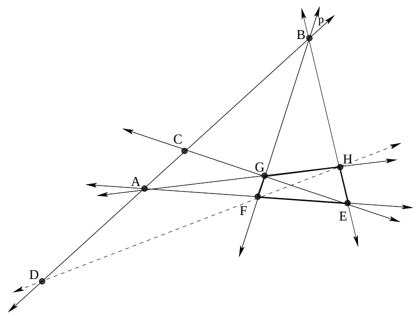
In the diagram above, we have constructed the harmonic set $H(OG, DB)$.

12. Answer the following questions based on the following diagram:

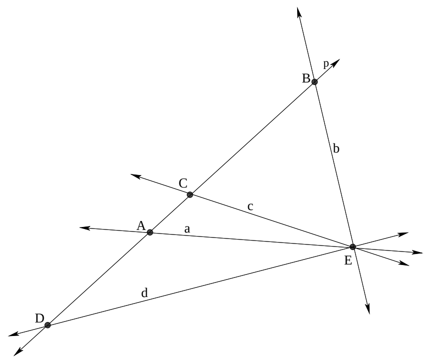


(a) Find D, the harmonic conjugate of C with respect to A and B.

To find the harmonic conjugate of C with respect to A and B, we construct am appropriate quadrangle (one with A and B as diagonal points and C the intersction of one of the remaining pair opposite sides) we then construct D to complete the harmonic set by finding the point that the remaining opposite side intersects the line $\overleftrightarrow{AB}$.

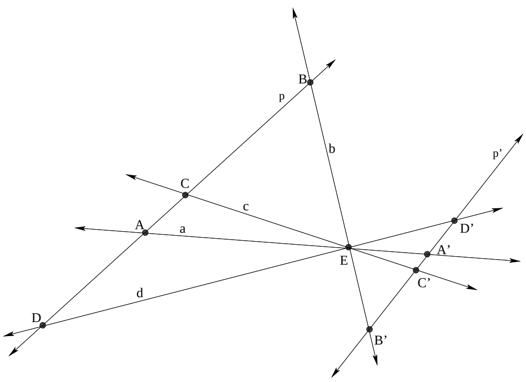


(b) Pick a point E not on $\overleftrightarrow{AB}$ and construct an elementary correspondence between the points A, B, C, D and a pencil of lines with center E.



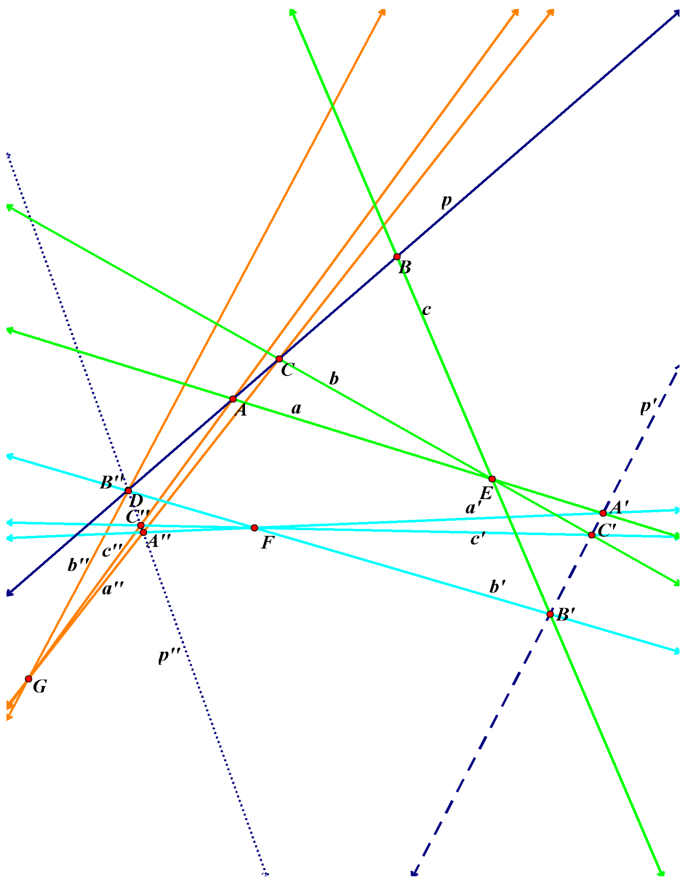
The diagram given above illustrates the elementary correspondance $ABCD \barwedge abcd$

(c) Find a line p' distinct from $p = \overleftrightarrow{AB}$ and extend the elementary correspondence you constructed in part (b) to a perspectivity between A, B, C, D and corresponding points on p'.



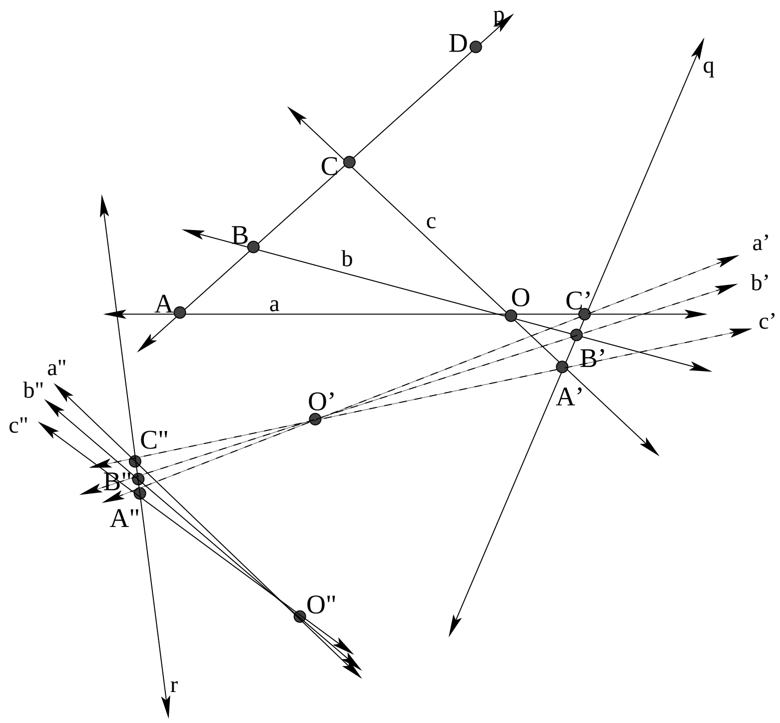
The diagram given above illustrates the perspectivity $ABCD \overset{p'}{\barwedge} A'B'C'D'$

(d) Extend this perspectivity to a projectivity $ABC \wedge CDA$.



The diagram shown above illustrates a projectivity $ABC \wedge CDA$.

13. Given the following projectivity:



(a) Identify each elementary correspondance in this projectivity.

The elementary correspondances are as follows:

$ABC \barwedge abc \barwedge A'B'C' \barwedge a'b'c' \barwedge A''B''C'' \barwedge a''b''c''$

(b) Find the image of D under this projectivity.

The image of the point d under this projectivity is the line d'' as illustrated in the following diagram: