Math 487
Exam 3 - Practice Problems

1. Consider the following functions:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + x$
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - 3$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x, y^2)$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x^3, y)$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (5x, 2y)$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (-x, -y)$

   (a) Which of these mappings are transformations? Justify your answer.
   (b) Which of these mappings are affine transformations? Justify your answer.
   (c) Which of these mappings are isometries? Justify your answer.

2. Prove each of the following:

   (a) The inverse of a transformation is a transformation.
   (b) The inverse of an isometry is an isometry.
   (c) The set of all transformations of a plane forms a group under the operation function composition.
   (d) Given f, an isometry of $\mathbb{E}$ and a triangle $\triangle ABC$, if $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$, then $\triangle ABC \cong \triangle A'B'C'$.

3. (a) Find homogeneous coordinates for the line $\ell[l_1, l_2, l_3]$ containing the points $(-3, 1, 1)$ and $(1, -2, 1)$.
   (b) Find the point of intersection of the lines $[4, -1, 0]$ and $[3, 2, -10]$.
   (c) Find the angle between the lines $[4, -1, 0]$ and $[3, 2, -10]$.

4. Define each of the following terms:

   (a) a transformation of a plane.
   (b) an isometry.
   (c) a group.
   (d) a reflection.
   (e) a translation.

5. (a) Find the matrix for the transformation T_{PQ} given:

      i. $P(2, 3, 1)$, $Q(5, 3, 1)$
      ii. $P(2, 3, 1)$, $Q(-1, 5, 1)$

   (b) Find the matrix for the transformation $R_{C,\theta}$ given:

      i. $C(0, 0, 1)$ and $\theta = 135°$
      ii. $C(-1, 2, 1)$ and $\theta = 30°$

   (c) Find the matrix for R_ℓ given:

      i. $\ell[100]$
      ii. $\ell[111]$

   (d) Find the matrix for G_{PQ} given $P(2, 3, 1)$ and $Q(3, 3, 1)$.

6. For each statement, determine whether the statement is true or false. Then briefly justify your answer.

   (a) Every isometry of $\mathbb{E}$ is a translation, a rotation, or a reflection.
   (b) The product of any two distinct rotations is a translation.
   (c) A nontrivial translation has no invariant points.

(d) A nontrivial rotation has exactly one invariant point.
(e) A nontrivial translation has no invariant lines.
(f) A nontrivial rotation has no invariant lines.
(g) A nontrivial reflection has exactly one invariant line.

7. Let f be the transformation given by the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

   (a) Find the image of the points $P(1, 3, 1)$ and $Q(-2, 5, 1)$ under this transformation.
   (b) Find the image of the line $[1 - 24]$ under this transformation.
   (c) What transformation is this?.

8. Given the points $P(2, 1, 1)$ and $Q(4, 2, 1)$

   (a) Find the matrix of a *translation* that maps P to Q.
   (b) Find the matrix of a *reflection* that maps P to Q.
   (c) Find the matrix of a *rotation* that maps P to Q.

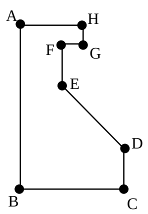
9. Consider the following transformation matrices:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -5 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad E = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

   (a) Which of these are the the matrix of an affine transformation of $\mathbb{E}$?
   (b) Which of these are the the matrix of an isometry of $\mathbb{E}$?
   (c) Which of these are the the matrix of an direct isometry of $\mathbb{E}$?
   (d) Which of these are the the matrix of a rotation of $\mathbb{E}$?
   (e) Which of these are the the matrix of a translation of $\mathbb{E}$?

10. Given the plane figure in $\mathbb{E}$ shown below, accurately draw the image of this figure under each of the following isometries:



   (a) $R_{B,90}$
   (b) T_{FG}
   (c) R_ℓ, where $\ell = \overleftrightarrow{HG}$.
   (d) R_ℓ, where $\ell = \overleftrightarrow{ED}$.
   (e) G_{CD}

11. Prove or Disprove:

   (a) The set of all *translations* of $\mathbb{E}$ forms a group under composition.
   (b) The set of all *rotations* of $\mathbb{E}$ forms a group under composition.
   (c) The set of all *indirect isometries* of $\mathbb{E}$ forms a group under composition.

12. Let $\square ABCD$ be the unit square centered at the origin in $\mathbb{E}$.

   (a) Give a complete list of all of the symmetries of $\square ABCD$.
   (b) Show that the set of isometries that you found in part (a) forms a group under composition.